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# THE PLASTIC EQUALIZATION METHOD FOR BENDING MOMENTS AND THE ROTATION CAPACITY OF PLASTIC HINGES IN CONTINUOUS REINFORCED CONCRETE BEAMS IN LIGHT OF EUROCODE REQUOREMENTS

***Marek Jędrzejczak*[1], *Michał Knauff*[2]**

[1] Department of Civil Engineering and Building Structures
University of Warmia and Mazury in Olsztyn
[2] The Institute of Building Structures
Warsaw University of Technology

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A b s t r a c t

The method worked out by JĘDRZEJCZAK and KNAUFF in 2007 has been used to calculate rotation in plastic hinges of the continuous reinforced concrete beams. It has been stated that when the degree of redistribution of bending moments does not exceed the allowable limits described in the Eurocode, the rotation does not exceed the limit values indicated in the norm either. Using the plastic equalization method for bending moments according to the Polish norm – without limiting the degree of redistribution – leads to excessive rotation.

## METODA PLASTYCZNEGO WYRÓWNANIA MOMENTÓW A ZDOLNOŚĆ DO OBROTU W PRZEGUBACH PLASTYCZNYCH ŻELBETOWYCH BELEK CIĄGŁYCH W ŚWIETLE WYMAGAŃ EUROKODU

***Marek Jędrzejczak*[1], *Michał Knauff*[2]**

[1] Katedra Budownictwa i Konstrukcji Budowlanych
Uniwersytet Warmińsko-Mazurski w Olsztynie
[2] Instytut Konstrukcji Budowlanych
Politechnika Warszawska

S ł o w a k l u c z o w e: budownictwo, konstrukcje żelbetowe, redystrybucja momentów zginających, przeguby plastyczne.

A b s t r a k t

Do obliczania kątów obrotu w przegubach plastycznych żelbetowych belek ciągłych zastosowano sposób opracowany przez JĘDRZEJCZAKA i KNAUFFA (2007). Stwierdzono, że jeżeli stopień redystrybucji nie przekracza dopuszczalnych granic określonych w Eurokodzie, to kąty obrotu również nie przekraczają wartości granicznych wskazanych w tej normie. Zastosowanie plastycznego wyrównania momentów według polskiej normy – bez ograniczania stopnia redystrybucji – wywołuje nadmierne kąty obrotu.

## Introduction

According to the Polish norm PN-B-03264:2002 it is possible to use several methods of nonlinear analysis of reinforced concrete construction. General principles of the plastic method of analysis and the elastic analysis with limited redistribution of moments in the Polish norm have been taken from Eurocode 2 (1992). Additionally, the Polish norm also contains the method of plastic equalization for moments in one way spanning slabs uniformly loaded and secondary continuous beams (point 9.1.2.3 in the norm). This method has been based on the 1970s editions of the norm.

The principles of the general, nonlinear method based on realistic models of reinforced concrete are not described in the Polish norm, but can be found in Eurocode 2 (from 1999 and 2008) and a manual (*Podstawy projektowania...* 2006).

The literature concerned with the application of the theory of plasticity to calculate concrete slabs, beams and frames is very extensive. The book by TICHY and RAKOSNIK (1971) contains 108 items listed in its bibliography. But the book of the greatest practical value in Poland was, for a long time, the one written by KOBIAK and STACHURSKI (part one issued in 1973, and part 2 issued in 1979). Both parts of the book, and the numerous articles included in it, did not, however, pay too much attention to the problem of rotation capacity in plastic hinges.

Due to the development in reinforced steel production and concrete technology, we now use materials that are much stronger but less ductile than the ones used at the time when the application of the theory of plasticity in reinforced concrete structures was rather rudimentary. This is why it is not wise to use methods assuming the redistribution of bending moments (e.g. in continuous reinforced concrete slabs and beams) in contemporary designs without checking rotation capacity in plastic hinges.

The Polish norm PN-B-03264:2002 states that if redistribution of moments calculated according to the linear elastic method is applied, it is necessary to provide the critical cross-sections with the rotation capacity, so that they can adapt to the predicted redistribution. The norm does not contain any information about the allowable plastic rotation (such information can be found in the Eurocodes from 1992, 1999 and 2008) or calculation methods for such rotations. The only criterion mentioned in the Polish norm PN-B-03264:2002 (quoted after the 1999 edition of the Eurocode) in point 4.4.2 are the conditions referring to the allowable degree of the moments' redistribution. Details concerning calculating the rotation and the allowable degree of redistribution changed with each subsequent edition of the Eurocode (1992, 1999, 2008), and, as we have already mentioned, the Polish norm permits the use of the method developed forty years ago. Thus, the set of the current rules is not really coherent.

The method of plastic equalization of bending moments is of great practical importance in Poland. This method may be considered a very specific case of the linear-elastic method with redistribution, in which the moments in spans and supports have been equalized. In our article (JĘDRZEJCZAK, KNAUFF 2002) showed that using this method, the conditions concerning the allowable degree of the redistribution of moments are often (but not always) fulfilled – and then it is not necessary to control the plastic rotation capacity. There is, however, a question if the rotation appearing as a result of the redistribution predicted with the use of this method do not exceed the values permitted by Eurocode 2 (2008). The answer to this question is the subject of this article.

## The calculation method of rotation

The rotation $\theta$ in a plastic hinge in a continuous beam may be calculated with a good approximation with the use of the method presented by (JĘDRZEJCZAK, KNAUFF 2007), by using the following formula

$$\theta = \frac{1}{12\alpha_W} \frac{1-\delta}{\delta} \frac{l}{d} W \tag{1}$$

where:

$\alpha_W$ – coefficient depending on static structure, i.e. the number of spans and the distribution of live load,

$\delta$ – the degree of redistribution, i.e. the ratio of the redistributed moment to the elastic bending moment,

$l/d$ – span slenderness ratio ($l$ – the length of span, $d$ – the effective depth),

$W$ – coefficient depending on the reinforcement ratio and the characteristics of concrete and steel.

Coefficient $\alpha_w$ is calculated using the formula:

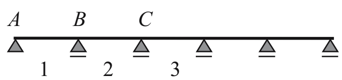
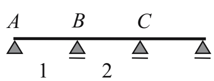
$$\alpha_W = \alpha_g \frac{g}{p} + \alpha_q \frac{q}{p} \tag{2}$$

where:
$g$ is dead load, $q$ – live, $p$ – total, $\alpha_g$ *and* $\alpha_q$ are coefficients to calculate extreme moments (e.g. according to Winkler's table).

Table 1 shows the values of $\alpha_W$ over supports B ($\alpha_g$ = –0.105, $\alpha_q$ = –0.119) and C ($\alpha_g$ = –0.079, $\alpha_q$ = –0.111) in a five-span beam and over support B in a three-span beam ($\alpha_g$ = –0.100, $\alpha_q$ = –0.117) for different ratios of loads $q/g$.

Table 1

Values of $\alpha_w$

| $q/g$ | 0.5 | 1.0 | 2.0 | 4.0 | 8.0 |
|---|---|---|---|---|---|
|  | A five-span beam | | | | |
| $\alpha_w$ (B) | 0.110 | 0.112 | 0.114 | 0.116 | 0.117 |
| $\alpha_w$ (C) | 0.090 | 0.095 | 0.100 | 0.105 | 0.107 |
|  | A three-span beam | | | | |
| $\alpha_w$ (B) | 0.106 | 0.109 | 0.111 | 0.114 | 0.115 |

Coefficient $W$ is calculated using the following formulae

$$W = \frac{\xi_{eff}(1 - 0.5\xi_{eff})\, f_{cd}}{E_{cd} j_{II}}, \quad j_{II} = \frac{J_{II}}{bd^3}, \quad \xi_{eff} = \rho \frac{f_{yd}}{f_{cd}} \tag{3}$$

where:
$J_{II}$ is the moment of inertia of the cross-section in a clear phase II. The values of $W$ are presented in diagrams a), b) and c) of Figure 1.

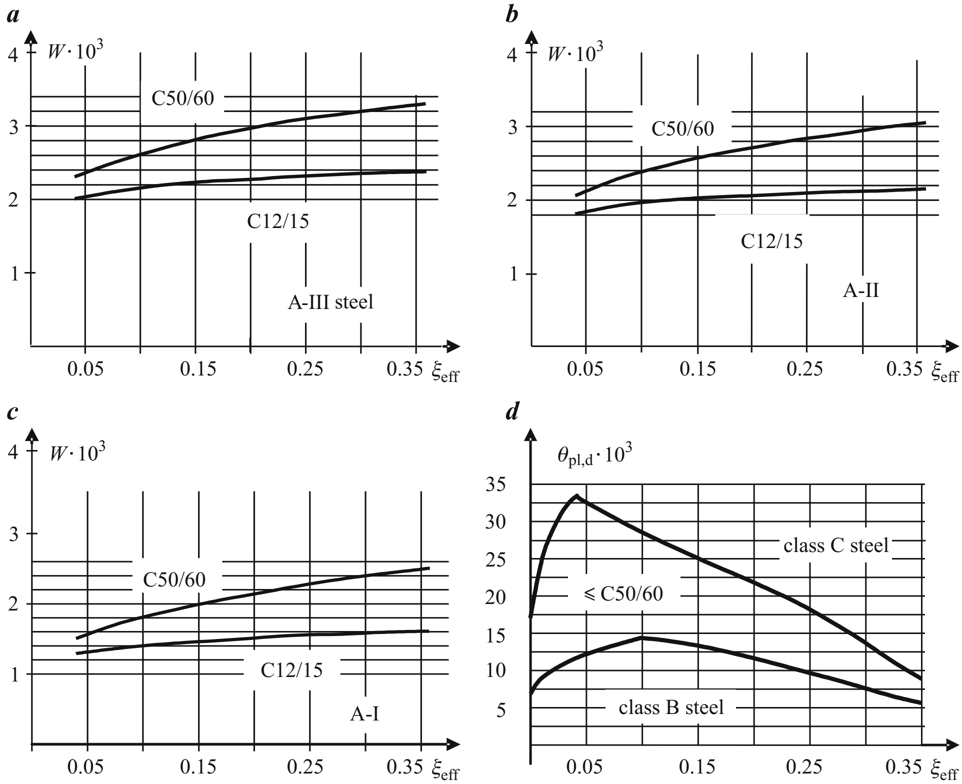



Fig. 1. Diagrams for checking the allowable plastic rotation – a, b, c) the values of expression *W*; d) allowable plastic rotation $\theta_{pl,d}$ based on fig. 5.6N in Eurocode 2 (2008); (steel classes which are described in PN-B-03264:2002 except A-IIIN fulfill the requirements appropriate for B ductility class according to the Eurocode 2 from 2008)

Figure 1 d) shows the allowable plastic rotation $\theta_{pl,d}$ based on figure. 5.6N in the Eurocode (2008) calculated for shear slenderness $\lambda = 3{,}0$. If the shear slenderness is different, then the value of allowable plastic rotation $\theta_{pl,d}$ taken from figure 1 d) has to be multiplied by $k_\lambda$ obtained from: $k_\lambda = \sqrt{\lambda/3}$, where $\lambda = a_q/d$, and $a_q$ is the distance between point zero of the moments diagram and the support. For the sake of simplification, it can be assumed that $\lambda = M_{Sd}/(Vd)$.

## Rotation in plastic hinges after equalizing the moments

In the method of plastic equalization, bending moments are calculated using the following formulae:

in the first and last spans and on the second and last but one supports:

$$M_1 = \frac{(g + q)}{11} l_{eff}^2 = 0.0909\, pl_{eff}^2 \tag{4}$$

in intermediate spans and in intermediate supports:

$$M_2 = \frac{(g + q)}{16} l_{eff}^2 = 0.0625\, pl_{eff}^2 \tag{5}$$

If live is equal to dead load (with different load ratios $q/g$, the values are different as well, but – as show in Table 1 – the influence of the $q/g$ ratio has no significant meaning), then the coefficients $\alpha_w$ and the degree of redistribution $\delta$ as well as the coefficients of shear slenderness for the five-span beams are: over the second and last but one supports (B):

$$\alpha_w = 0.112,\ \delta = \frac{M_R}{M_e(p)} = \frac{0.0909 pl_{eff}^2}{\alpha_w pl_{eff}^2} = \frac{0.0909}{0.112} = 0.8116$$

$$\lambda = \frac{M_R}{Vd} = \frac{0.0909 pl^2}{0.5 pld} = 0.1818 \frac{l}{d}$$

over the intermediate support (C):

$$\alpha_w = 0.095,\ \delta = \frac{0.0625 pl_{eff}^2}{\alpha_w pl_{eff}^2} = \frac{0.0625}{0.095} = 0.6579$$

$$\lambda = \frac{0.0625 pl^2}{0.5 pld} = 0.125 \frac{l}{d}$$

The corresponding values for the three-span beams are:

$$\alpha_w = 0.109,\ \delta = \frac{0.0909}{0.109} = 0.8339,\ \lambda = 0.1818 \frac{l}{d}$$

Table 2 shows rotation over supports B and C in the five-span beam and over support B in the three-span beams, as well as the allowable plastic rotation according to the Eurocode (2008) for four values of $\xi_{eff}$ and four slenderness ratios of the span $l/d$. Calculations have been done for B25 concrete and A-III steel. Table 3 shows a sample calculation for support B of a five-span beam, when $\xi_{eff} = 0.05$ and $\lambda = 2.0$ (formulae according to *Podstawy projektowania...* 2006).

Table 2

Rotation over supports

| $\xi_{eff}$ | $\lambda$ | Three-span beam $\theta_B$ [mrad] | Five-span beam | | $k_\lambda \cdot \theta_{pl,d}$ [mrad] |
|---|---|---|---|---|---|
| | | | $\theta_B$ [mrad] | $\theta_C$ [mrad] | |
| (1) | (2) | (3) | (4) | (5) | (6) |
| 0.05 | 1.0 | 1.80 | 2.03 | 7.82 | 6.70 |
| | 2.0 | 3.59 | 4.06 | 15.65 | 9.47 |
| | 3.0 | 5.39 | 6.10 | 23.47 | 11.60 |
| | 4.0 | 7.19 | 8.13 | 31.30 | 13.39 |
| 0.10 | 1.0 | 1.92 | 2.17 | 8.36 | 8.02 |
| | 2.0 | 3.84 | 4.34 | 16.73 | 11.34 |
| | 3.0 | 5.76 | 6.52 | 25.09 | 13.90 |
| | 4.0 | 7.68 | 8.69 | 33.46 | 16.05 |
| 0.15 | 1.0 | 2.00 | 2.27 | 8.72 | 7.51 |
| | 2.0 | 4.00 | 4.53 | 17.43 | 10.61 |
| | 3.0 | 6.01 | 6.81 | 26.15 | 13.00 |
| | 4.0 | 8.01 | 9.07 | 34.87 | 15.01 |
| 0.20 | 1.0 | 2.07 | 2.34 | 9.01 | 6.64 |
| | 2.0 | 4.14 | 4.68 | 18.02 | 9.39 |
| | 3.0 | 6.21 | 7.02 | 27.03 | 11.50 |
| | 4.0 | 8.27 | 9.36 | 36.04 | 13.28 |
| 0.25 | 1.0 | 2.12 | 2.39 | 9.21 | 5.66 |
| | 2.0 | 4.23 | 4.79 | 18.43 | 8.00 |
| | 3.0 | 6.35 | 7.18 | 27.64 | 9.80 |
| | 4.0 | 8.46 | 9.57 | 36.85 | 11.32 |
| 0.30 | 1.0 | 2.15 | 2.43 | 9.37 | 4.56 |
| | 2.0 | 4.30 | 4.87 | 18.73 | 6.45 |
| | 3.0 | 6.45 | 7.30 | 28.10 | 7.90 |
| | 4.0 | 8.60 | 9.37 | 37.47 | 9.12 |
| 0.35 | 1.0 | 2.18 | 2.46 | 9.48 | 3.46 |
| | 2.0 | 4.35 | 4.92 | 18.95 | 4.90 |
| | 3.0 | 6.53 | 7.39 | 28.43 | 6.00 |
| | 4.0 | 8.70 | 9.85 | 37.90 | 6.93 |

Table 3

Sample calculation

| | |
|---|---|
| $E_{cd} = \frac{E_{cm}}{\gamma_c}$ | $E_{cd} = \frac{30}{1.2} = 25GPa$ |
| $\alpha_e = \frac{E_s}{E_{cd}}$ | $\alpha_e = \frac{200}{25} = 8$ |
| $\rho = \xi_{eff}\frac{f_{cd}}{f_{yd}}$ | $\rho = 0.05\,\frac{13.3}{350} = 0.0019$ |
| $\alpha_1 = \alpha_e\rho$ | $\alpha_1 = 8 \cdot 0.0019 = 0.0152$ |
| $\xi_{II} = \sqrt{\alpha_1^2 + 2\alpha_1} - \alpha_1$ | $\xi_{II} = \sqrt{0.0152^2 + 2 \cdot 0.0152} - 0.0152 = 0.1598$ |
| $j_{II} = \frac{\xi_{II}^3}{3} + \alpha_1\,(1 - \xi_{II})^2$ | $j_{II} = \frac{0.1598^3}{3} + 0.0152 \cdot (1 - 0.1598)^2 = 0.0121$ |
| $W = \frac{\xi_{eff}\,(1 - 0.5\xi_{eff})}{j_{II}}\,\frac{f_{cd}}{E_{cd}}$ | $W = \frac{0.05(1 - 0.5 \cdot 0.05)}{0.0121}\,\frac{13.3}{25 \cdot 10^3} = 2.145 \cdot 10^{-3}$ |
| $\lambda = \frac{M_R}{Vd} = \frac{0.0909p_Rl^2}{0.5p_Rld} = 0.1818\,\frac{l}{d} \Rightarrow \frac{l}{d} = 5.5\lambda = 5.5 \cdot 2 = 11$ | |
| $k_\lambda = \sqrt{\frac{\lambda}{3}}$ | $k_\lambda = \sqrt{\frac{2}{3}} = 0.816$ |
| $\theta = \frac{1}{12\alpha_w}\,\frac{1 - \delta}{\delta}\,\frac{l}{d}\,W \leq k_\lambda\theta_{pl,d}$ | $\theta = \frac{1}{12 \cdot 0.112}\,\frac{1 - 0.8116}{0.8116} \cdot 11 \cdot 2.145 \cdot 10^{-3} = 4.06 \cdot 10^{-3} \leq$ <br> $\leq 0.816 \cdot 11.6 \cdot 10^{-3} = 9.47 \cdot 10^{-3}$ |

## Conclusion

As shown in the analysis (results presented in Table 2), while using the plastic equalization of bending moments, rotation in plastic hinges, which may then appear, are unacceptable according to Eurocode 2 (2008) (i.e. the rotation in columns 3, 4 or 5 are bigger than in column 6 of Table 2). Thus, this method – even though the norm PN-B-03264:2002 presents no objections to it – cannot be used without checking the rotation in plastic hinges.

Comparing the results with the conclusion of the article by JĘDRZEJCZAK, KNAUFF (2002), we have to state that the simple conditions offered in the norm PN-B-03264:2002, which exempt us from calculating the rotation in plastic hinges (with regard to the linear-elastic method with redistribution), are also appropriate for the method of plastic equalization for bending moments, because the values of rotation presented in Table 2 are exceeded only when the norm PN-B-03264:2002 does not exempt us from the obligation to control these rotations. Exceeding the allowable plastic rotations appears, for example, in a three-span beam with significant reinforcement ($\xi_{eff}$ = 0.35, when $\lambda \geq 3$) and in a five-span beam over the central support C, when $q/g > 0.5$.

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