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SELECTED PROBLEMS OF MEASUREMENT UNCERTAINTY – PART 1

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Abstract

The rules for evaluation of measurement uncertainty are presented in this paper. It includes the list of definitions for basic terms that are associated with the issue and explains how to evaluate the measurement uncertainty for well-defined physical parameters that are considered as the measurands (part 1). Next, the methods for evaluation of measurement uncertainty are exhibited on the examples when measurement uncertainty is estimated for verification of a micrometer as well as evaluated for basic reliability parameters (part 2).

WYBRANE ZAGADNIENIA NIEPEWNOŚCI POMIARU – CZ. 1

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Słowa kluczowe: niepewność pomiaru, niepewność typu A i typu B, błąd pomiaru, rozkład normalny i prostokątny, budżet niepewności.

Abstrakt

W pracy podano zasady postępowania podczas wyznaczania niepewności pomiaru. Przedstawiono podstawowe pojęcia dotyczące tego zagadnienia oraz sposób wyznaczania niepewności pomiaru dobrze określonej wielkości fizycznej stanowiącej wielkość mierzoną (cz.1). Przedstawiono sposób szacowania niepewności pomiaru na przykładzie szacowania niepewności podczas sprawdzania mikrometru i szacowania niepewności podstawowych parametrów wytrzymałościowych (cz. 2).

Introduction

Every measurement is aimed at determination of the measured value, i.e. value of the specific parameter that is the objective of measurements.

Yet measurements results are only approximated or evaluated (estimated) value of the measured parameter and therefore the results is complete only when it is provided with the estimated uncertainty.

Following the Manual Expression of Measurement Uncertainties (*Wyrażanie niepewności pomiaru*. 1999):

"Uncertainty (of measurement) is the parameter that is associated with measurement results and exhibits dispersion of values that can be assigned to the measured value in justified manner".

Expression of uncertainty in measurement refers to a well-defined physical parameter that is considered as the measured value and can be characterized by z single value obtained from measurements. The term "value" (measurable quantity) stands for the property of the phenomenon, material or substance that can be qualitatively described and quantitatively evaluated. Uncertainty of measurements is a result of random errors that can occur during measurements. It is important to distinguish the term "error" and the term "uncertainty of measurements". Every error represents a random variable, while uncertainty of measurements is a parameter for distribution of error probabilities.

In accordance to requirements of the standard PN-EN ISO 10012 (PN-EN ISO 10012:2004):

– evaluation of the uncertainty should take account for not only uncertainty of the measurement equipment calibration, but also for all the other components of uncertainty that are essential for the specific measurement process. Relevant methods for the analysis should be applied,

– if any uncertainty components are so insignificant as compared to other components that evaluation of such insignificant components is unjustified in technical and economical aspects, calculation thereof should be given up and the decision should be recorded,

– measurement uncertainty should be evaluated for every measurement process that is covered by any measurement management system,

– measurement uncertainty should be recorded and all the known reasons for measurement variations must be documented,

– efforts devoted to evaluation and recording of measurement uncertainty should be commeasurable to the significance of the measurement results to the product quality.

In accordance to requirements of the standard PN-EN ISO/IEC 17025, the accredited laboratory:

– it should have in place the working procedure on evaluation of measurement uncertainty,

– when it performs its own calibration and verification procedures it should have in place the working procedure (instruction) for evaluation of measurement uncertainty for every calibration or verification,

– when the character of the applied research method disables strict, metrological and statistically justifiable calculations of measurement uncertainty the efforts should be made to identify all the uncertainty components and estimate them in reasonable manner based on available knowledge on capabilities of the method and gathered past experience, such a way of uncertainty evaluation should de described,

– in case when examinations are carried out by means of a well-known method where boundary limits for main sources of measurement uncertainty and the methods for result reporting are truthfully defined, such results should be reported in accordance with the relevant statements of the research procedure or instruction,

– sources of uncertainties include the applied patters and reference materials, used methods and equipment, environmental conditions, properties and status of the objects that are subject to examination or calibration as well as the performing staff.

Measurement equation

The dependence of measurement results, considered as the random variable Y, on a number of random variables X_1, ..., X_n, associated with the measurement process, i.e. directly measurable parameters, correction factors, physical constants and deviations and errors thereof is described by the formula:

$$Y = f(X_1, X_2, X_3, ..., X_{n-1}, X_n) \quad (1)$$

The function f of the above equation may express no physical law and merely describe the measurement process. It should cover all the parameters that can affect the modelling of the measurement result. Initial parameters (arguments for the function f) should be defined as accurately as practically feasible in order to determine values thereof in unambiguous manner.

The estimation of the initial parameter Y, that is adopted as the measurement result, shall be the value of y that is determined by the same equation as above, but with the X_1, ..., X_n values substituted with their estimators x_1, ..., x_n, namely:

$$y = f(x_1, x_2, x_3, ..., x_{n-1}, x_n) \quad (2)$$

Determining of standard uncertainties for all the components

At the outset, it is necessary to estimate standard uncertainties $u(x_i)$ for all the initial parameters that influent the research process (Dokument EA-4/02 1999, PIOTROWSKI, KOSTYRKO 2000, PN-EN ISO/IEC 17025 2005). For the research (measurements) where series of observations are carried out the **method** of statistical analysis of the **A-type** is applicable to the series of observations. If uncertainty of the input value cannot be determined on the basis of a series of measurements, the **B-type method** is applied, where the standard uncertainty is evaluated on the basis of information about possible range of variations for the measured parameter in question.

The A-type method (for the measured value or errors of its measurements described by Gaussian distribution of measurement results) achieves the best evaluation (estimator) for the mathematical expectation μ associated with the random variable x as the arithmetical mean $\bar{x}$ provided that n independent observations were made under repeatable measurement conditions.

$$\bar{x} = \frac{1}{n} \sum_{1}^{n} x_i \tag{3}$$

The standard uncertainty of the A-type is equivalent to standard deviation of the mean value over a series of experiments. The standard deviation is calculated by the following formula:

$$u_A\ (\bar{x}) = \frac{u(x)}{\sqrt{n}} \tag{4}$$

where:

$$u(x) = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2} \tag{5}$$

The foregoing formulas are true only for sufficiently long series of observations ($n > 30$ is usually assumed), when the mean value is a trustworthy estimation for the mathematical expectation.

The correction factor $\frac{1}{\sqrt{n}}$ for the experimental standard uncertainty drops rapidly nearby the beginning of the coordinate system (for $n = 4$ it is halved) but for large n (above a dozen) tends to decrease rather slowly. It is why excessive expanding of measurement series seems to be unjustified. Needless to say that it is really difficult to make sure that measurement conditions shall be repeatable for long series of measurements. However, number of measurements should be high enough to guarantee that the arithmetical mean is a trustworthy estimation for the mathematical expectation.

Long series of measurements ($n > 30$) should be launched when the combined standard uncertainty $u_p(x)$ is to be found out. Such a type of uncertainty is used later on when similar measurements are taken under the same conditions. Then, after completion of the series of n_1 new measurements, the variation of measurement (observation) distribution becomes known and the uncertainty can be determined by the following formula:

$$u_A\ (\bar{x}) = \frac{u_p(x)}{\sqrt{n_1}} \tag{6}$$

If variation of measurement (observation) distribution remains unknown, the required number of measurements in a series depends on the desired confidence interval for the expanded uncertainty.

In case of the B-type method, where we have to deal with a single measurement result or with figures acquired from documents or literature references, the boundary limits a_+ and a_- can be evaluated for the input variable X_i and then the standard uncertainty is calculated by the formula:

$$u_B\ (x_i) = \frac{a}{\sqrt{k}} \tag{7}$$

where:

$$a = \frac{a_+ + a_-}{2}$$

The value $u_B(x_i)$ as calculated in the foregoing manner is referred to as the standard uncertainty of B-type. It is the method for analysis of conditions where underlying reasons for errors may occur and is recommended for analysis and estimation of instrumental (equipment) errors.

The following features attributable to any source of errors should be taken into account:

– availability of information on the probability distribution applicable to the specific parameter (e.g. Gaussian, rectangular, triangular),

– information on possible boundaries of the variability interval for the specific parameter value, x_i belongs to (a_-, a_+),

– probability of the fact that the specific value of the x_i parameter falls into the predefined interval.

Calculation results, obtained in the above manner do not have to be worse than those that are based on repeatable observations. Table 1 presents the values of "uncertainties of uncertainties" that origin in the statistic way, depending on the number of observations. The data in the table exhibit that even for $n = 10$ observations the "uncertainty of uncertainties" is still as high as 24%. This allows to conclude that calculation of standard uncertainty with use of the A-type method do not have to be more dependable than in case when the standard uncertainty is calculated with use of the B-type method and in many cases of experiments when number of measurements must be limited, the components obtained from calculations with use of the B-type method may be even better defined than respective components that are obtained from calculations with the A-type method.

Table 1

Ratios of experimental standard deviations for the arithmetical means for n independent observation with Gaussian distribution over standard deviation of the mean value

Number of observations n	$\sigma[s(\bar{q})] / \sigma(\bar{q})$ %
2	76
3	52
4	42
5	36
10	24
20	16
30	13
50	10

Sourece: Dokument EA-4/02 (1999)

Determining of the composed standard uncertainty

For the values of parameters that are measured in direct way, when the standard uncertainties of both A and B types are taken into account, the composed standard uncertainty u_C is the square root of the sum of squares for the both uncertainty values.

$$u_C = \sqrt{u_A^2 + u_B^2} \tag{8}$$

In most cases the parameter value y in not measured in direct way but determined on the basis of measurements for other parameters x_i that remain in strict relations with the desired parameter that are defined by a specific function (see measurement equation (6)).

Based on the total differential the *law for propagation of uncertainties* is formulated in the following way:

$$u_C(y) = \sqrt{\sum_{1}^{n}\left(\frac{\partial f}{\partial x_i}\right)^2 u^2(x_i)} \quad (9)$$

$$u(y_i) = |c_i| \cdot u(x_i) \quad (10)$$

where:
$u(x_i)$ – standard uncertainties for measurements of input parameters, calculated with use of either A or B method; the composed standard uncertainty $u_C(y)$ is then the estimation of the standard deviation σ_y and characterizes dispersion of values that by justified reasons can be associated with the measured value y,
$\frac{\partial f}{\partial x_i} = c_i$ – partial derivatives that are referred to as sensitivity coefficients.

The law for propagation of uncertainties is true in the input parameters are uncorrelated, which is the most common practice.

However, if the provision on mutual independence of input values is not met, the formula with account for covariances must be applied:

$$u_C^2(y) = \sum_{i=1}^{m}\left(\frac{\partial f}{\partial x_i}\right)^2 u^2(x_i) + 2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\left(\frac{\partial f}{\partial x_i}\right)\left(\frac{\partial f}{\partial x_j}\right) u(x_i, x_j) \quad (11)$$

The final result for measurements of the parameter Y is calculated on the basis of the defined function (measurement equation) where arithmetical means of directly measurable parameters are substituted:

$$\bar{y} = f(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n \quad (12)$$

Uncertainty budget

If individual input parameters are mutually independent, the information that is essential for analysis of the measurement uncertainty can be brought together into a table (see Table 2). Such a table assures clarity of the analysis and is referred to as the uncertainty budget.

Table 2
Uncertainty budget for calculation of a composed uncertainty for uncorrelated input parameters

Parameter symbol X_i	Parameter estimation x_i	Standard uncertainty $u(x_i)$	Probability distribution	Sensitivity coefficient c_i	Contribution to the composed uncertainty $u_i(y)$
X_1	x_1	$u_A(x_1) = \frac{U}{2}$	Gaussian	c_1	$u_1(y) = c_1 \cdot u_A(x_1)$
X_2	x_2	$u_B(x_2) = \frac{x_2}{2\sqrt{3}}$	rectangular	c_2	$u_2(y) = c_2 \cdot u_B(x_2)$
X_N	x_N	$u_B(x_N) = \frac{x_N}{2\sqrt{6}}$	triangular	c_N	$u_N(y) = c_N \cdot u_B(x_N)$
Y	y	–	–	–	$u_c(y)$

Source: PN-EN ISO 9000 (2001)

Determining of the expanded uncertainty

In case of direct measurements the expanded uncertainty U is the product of the expansion coefficient and the composed standard uncertainty:

$$U(y) = k_\alpha \cdot u_C(y) \qquad (13)$$

where $u_C(y)$ is calculated with use of the formula for direct measurements.

The expansion coefficient for the guaranteed confidence level p_α should be calculated on the basis of the distribution for a standardized random value, where the distribution is a convolution of both the Gaussian and uniform (rectangular) distributions, when the set of samples is large. Otherwise, when the set of sample size is small, the *t*-Student distribution should be applied.

For indirect measurements, the expanded uncertainty, denoted as U, is obtained by multiplication of the standard composed uncertainty $u_C(y)$ by the expansion coefficient k_α:

$$U(y) = k_\alpha \cdot u_C(y) \qquad (14)$$

where $u_C(y)$ is calculated with use of the formula for indirect measurements and the result in the following form:

$$y = y \pm U(y) \qquad (15)$$

is recorded for the confidence level of p_α.

Exact calculation of expansion coefficients for the desired confidence levels is a sophisticated job for indirect measurements as it needs to know the function of the probability density distribution for the random value that is used for modelling of measurement results y. Such a function is a convolution of component distribution for random variables that are used for modelling of input parameters. Calculation of such convolutions is difficult, except for some specific cases that include convolution of any number of Gaussian distributions, which is the Gaussian distribution itself and its parameters can be easily calculated.

It is why approximated methods are practically used for calculation of the expansion coefficient.

Approximated methods for calculation of the expanded uncertainty (*Analiza błędów...* 2007)

Method I – with use of predefined values for expansion coefficients

It consists in application of the coefficient $k_\alpha = 2$ for the confidence level $p_\alpha \approx 95\%$ and $k_\alpha = 3$ for the confidence level $p_\alpha \approx 99\%$.

It is the method that is used for measurement experiments that involve independent input parameters X_i and:

– distributions for all the components of the standard composed uncertainty $u_C(y)$ are of Gaussian type,

– distributions for the components of the standard composed uncertainty $u_C(y)$ are of rectangular type, but with the same width and not less than 3 distributions are available,

– there is quite many input parameters X_i (practically not less than 4), the composed standard uncertainty $u_C(y)$ is free of domination by the component of the standard uncertainty that is calculated by means of the A-type method for the series of only few observations or by the component of the standard uncertainty that is calculated by means of the B-type method from the presumed rectangular distribution (i.e. the uncertainties $c_i u(x_i)$ and $c_i u_B(x_i)$ contribute to the $u_C(y)$ i $u_C(y)$ in comparable degree) and is much higher than a single component $c_i u_B(x_i)$.

Method II – geometrical superposition

It consists in calculation of the expanded uncertainty U for every single input parameter (for every component of the error) and then the expanded uncertainty for the output value U_y is calculated as the square root of the sum of squares for all the component uncertainties, in accordance with the formula:

$$U_y = \sqrt{U_{x_1}^2 + U_{x_2}^2 + ... + U_{x_n}^2} \tag{16}$$

Expansion coefficients for component uncertainties must be calculated for the same confidence level.

Method III – ordinary (algebraic) sum

Adding in the uncertainty domain:

$$U = U_A + U_B \quad \text{or} \quad U_y = U_{x1} + U_{x2} + ... + U_{xn} \tag{17}$$

The method assumes the worst case for accumulation of errors, i.e. the situation when all the component errors are of the maximum value and are of the same sign. It is very little probably, which means that the method exaggerates uncertainty of measurements and is the most pessimistic one.

However, the method is used for workshop measurements due to easy calculation as well as for specific cases or, for instance, when tolerances are to be fund out or when components of machinery must be manufactured with defined play.

Method IV – dominating component

It is recommended for the cases when one of the component uncertainties of either A or B types is the dominating factor.

If $u_A \geq u_B$, the substitutions $k_\alpha = k_A$ must be made in the formula (13), Similarly, if $u_B \geq u_A$ the substitution in the formula (13) should be $k_\alpha = k_B$.

For the Gaussian probability distribution and for the confidence level of 95.45% the expansion factor should be $k_\alpha = 2$, whereas for the confidence level of 99.73% – $k_\alpha = 3$.

For the rectangular probability distribution and for the confidence level of 95% the expansion factor should be $k_\alpha = 1.65$, whereas for the confidence level of 99% – $k_\alpha = 1.71$.

Method V – effective degrees of freedom

For this method the expanded uncertainty is denoted as U_p and calculated by the following formula:

$$U_p = k_p \cdot u_C(y) = t_p(\nu_{\text{eff}}) \cdot u_C(y) \tag{18}$$

where t_p stands for the t coefficient attributable to the t-Student distribution that can be read from relevant reference tables in accordance with the selected confidence level p_α and for the effective number of the degrees of freedom ν_{eff}. The figure ν_{eff} is to be calculated by the Welch-Satterthwaite formula

$$\nu_{eff} = \frac{u_{\mathrm{C}}^4(y)}{\sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i}\right)^4 \frac{u_i^4(x_i)}{\nu_i}} \tag{18}$$

where:
$u_C(y)$ – composed standard uncertainty for the output value,
$u(x_i)$ – standard uncertainties for the input parameters, $i = 1,2, ..., N$,
ν_i – number of the degrees of freedom for $u(x_i)$.

Practical calculation of the number of degrees of freedom for $u(x_i)$:
- if distribution for the $u(x_i)$ is of the Gaussian type, then

$$\nu_i = n - 1$$

- if distribution for the $u(x_i)$ is of the rectangular type with the presumed width of a, then $u(x_i) = \frac{a}{2\sqrt{3}}$ is adopted with no uncertainty as the limits for such decomposition are known and then $\nu_i \to \infty$, hence $1/\ \nu_i \to 0$,
- if the source of errors is rated to the B group and available information of that error source indicate that the standard uncertainty $u(x_i)$ can be merely estimated with the specific relative error $\delta_{ui} = \frac{\Delta u(x_i)}{u(x_i)}$, then $\nu_i \approx \frac{1}{2} \cdot \frac{1}{\delta_{ui}^2}$.

For instance, under the assumption that $u(x_i)$ is estimated with the error of 25%, number of the degrees of freedom shall amount to: $\nu_i \approx 1/2 \cdot (1/0.25^2) \approx 8$.

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