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NUMERICAL ASPECTS OF A PNEUMATIC TYRE MODEL ANALYSIS

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Key words: tyre, computational code, finite element method, block diagram, deformation, strength.

Abstract

This paper presents a method of axisymmetric analysis of a pneumatic tyre model, based on the finite element method. A brief characterisation has been provided of a heterogeneous tyre model, which consists of three groups of materials: cord-rubber composite, physically nonlinear rubber and bead wire steel. The system is also geometrically nonlinear. This is all reflected in a detailed block diagram of the author's in-house code intended for analysis of deformation and strength of pneumatic tyres. The results of exemplary calculations made with the use of this code are presented.

NUMERYCZNE ASPEKTY ANALIZY MODELU OPONY PNEUMATYCZNEJ

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Słowa kluczowe: opona, program obliczeniowy, metoda elementów skończonych, schemat blokowy, deformacja, wytrzymałość.

Abstrakt

W pracy przedstawiono sposób analizy obrotowo-symetrycznego modelu opony pneumatycznej bazującego na metodzie elementów skończonych. Podano krótką charakterystykę modelu niejednorodnej opony, w której wyróżniono trzy grupy materiałów: kompozyt kord-guma, fizykalnie nieliniową gumę i stal drutówek. Układ charakteryzuje się również nieliniowością geometryczną. Wszystko to ma swoje odbicie w szczegółowo przedstawionym schemacie blokowym programu autorskiego przeznaczonego do analizy deformacji i wytrzymałości opon pneumatycznych. Zamieszczono wyniki przykładowych obliczeń wykonanych za pomocą tego programu.

Introduction

Currently, the production of new commercial models of pneumatic tyres does not start until its geometric parameters and dimensions have been tested, the tyre has been mounted onto a rim and inflated, its strength, shape and distribution of pressure in the area of static contact with the ground have been examined and even the behaviour of the tyre under operation conditions has been checked. A future product is therefore tested and improved at the design phase, which requires a reliable computer model. Due to a considerable progress in computer methods in solid mechanics and power of computers, a number of commercial FEM codes are currently available which provide extensive analytic capabilities for even the most complex problems of the mechanics of a deformable body. Hence, a problem arises as to whether an analysis of deformation and strength of a pneumatic tyre can be performed directly with an FEM code, even if it is one of the most advanced software code. It is possible with one tyre; however, this is unacceptable to tyre designers due to the time needed to prepare a computational model. The fundamental difficulty lies in the fact that due to the technology of tyre formation, the density and angle of cords depend on the distance of a finite element from a tyre axis of rotation. For example, the cord density in the ply of a truck tyre decreases by ca. 50% between the tyre bead to the apex (PELC 1995c). Therefore, when preparing data which describe the mechanical properties of a tyre material, one should specify them separately for each finite element. This is why the author has spent years developing a tyre computational model based on the finite element method (PELC 1992, 1995a, 2000, 2002), followed by algorithms and a computer code intended for analysis of pneumatic tyre deformation analysis, in which data are generated from the properties of a raw layer of cord-rubber composite (wound on a tyre building drum) and the position of an element in a formed tyre. Moreover, a tyre is characterised by complex geometry and heterogeneity.

Loads acting on a tyre may cause considerable displacements and some materials, such as rubber, behave as nonlinear-elastic elements. Therefore, a tyre model is a nonlinear model and determination of an equilibrium path for such a system requires an incremental-iterative technique. All these issues are reflected in a block diagram of the author's code, intended for pneumatic tyre model axisymmetric analysis.

Characterisation of an axisymmetric tyre model

A pneumatic tyre has a complex structure and must be regarded as a heterogeneous system (Fig. 1). Three types of materials can be distinguished in a tyre:
– cord-rubber composite (ply layers, belts and chafer),

– rubber (e.g. tread, sidewall, etc.),
– steel (bead wire).

In the case of simulation of a tyre mounting on a rim or inflation, one can assume that a tyre is an axisymmetric system, *i.e.* that its points are not moving circumferentially. In such a case, only a cross section of a tyre can be considered (Fig. 2a) and only this area undergoes discretization. Although the finite elements seem to be plane, in fact they are rings with a triangular or rectangular cross-section.

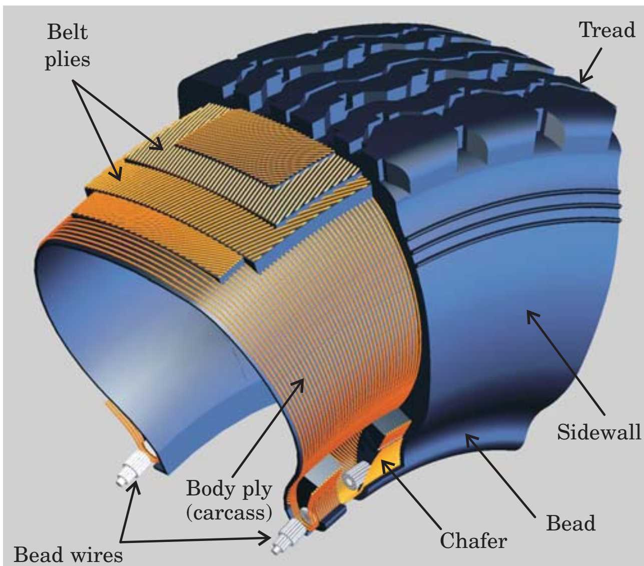


Fig. 1. The structure of a pneumatic radial truck tyre

Unlike a plane problem (2D), this one is classified as a 2.5D problem, *i.e.* there are non-zero circumferential stresses. The finite element mesh, fixing of the tyre bead and the action of internal pressure are presented in Fig. 2b. Nodes situated in the tyre plane of symmetry (*yz*) can move only in the radial direction.

The angle of the cord to the parallel line of the layer in a formed tyre (θ) can be calculated from the so-called "panthograph role" (HOFFERBERTH 1956):

$$\cos\ \theta = \frac{r}{r_0} \cos\ \theta_0 \tag{1}$$

where:
r_0/r – the radius of a point of a layer on a drum/in a tyre,
θ_0 – angle of a cord in a green layer.

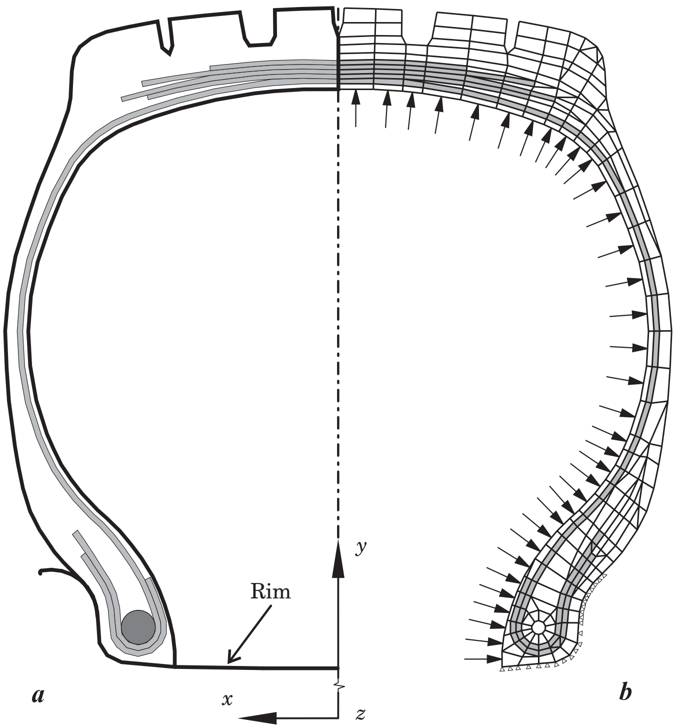


Fig. 2. An axisymmetric model of a truck tyre: *a* – cross section with layers, *b* – mesh of finite elements and boundary conditions

The cord density in a tyre layer depends on the density in a green layer placeed on a drum e_{p_0}:

$$e_p = e_{p_0} \frac{r_0 \sin \theta_0}{r \sin \theta} \tag{2}$$

and it allow to determine the ply cord volume fraction based on the cross section area of a cord (A_c) and the layer thickness (t):

$$v_c = \frac{e_p A_c}{t} \tag{3}$$

The fraction value provides the basis for calculation of effective material constants for a cord-rubber composite layer. Assuming that the shear modulus for cords is greater than it is for rubber, HALPIN and TSAI (1969) formula yields:

$$\begin{aligned} E_1 &= E_c v_c + E_r (1 - v_c) \\ E_2 &= \frac{E_r(1 + 2v_c)}{1 - v_c} \\ \nu_{12} &= \nu_c v_c + \nu_r(1 - v_c) \\ G_{12} &= G_r \frac{1 + v_c}{1 - v_c} \end{aligned} \tag{4}$$

where:
letters $E/G/\nu$ – denote Young and Kirchhoff moduli and Poisson ratio, respectively, and the c/r indices denote cord and rubber, respectively. Axis 1 is parallel to cord, while axis 2 denotes one that is perpendicular to it, lying on the layer plane.

In a displacement formulation of FEM, assumed in the axisymmetric tyre model, it is convenient to describe rubber properties as if it was a nearly incompressible material, as proposed by BLATZ and KO (1962).

Having determined the tangent constitutive tensor for the material (${}_0\mathbf{D}$), one can calculate the second Piola-Kirchhoff stress tensor (${}_0\mathbf{S}$) based on the known increment of Green-Lagrange strain tensor (${}_0\mathbf{E}$):

$${}_0\mathbf{S} = {}_0\mathbf{D}\,{}_0\mathbf{E} \tag{5}$$

Defining the tensor as (BROCKMAN 1986)

$${}_0^t\mathbf{H} = \frac{\partial I_3}{\partial\, {}_0^t\mathbf{C}} \tag{6}$$

where:
${}^t\mathbf{C}$ denotes Cauchy-Green deformation tensor, and I_3 – its third invariant, and taking into account the strain energy density function in the Blatz-Ko model for a nearly incompressible material, the second Piola-Kirchhoff stress tensor can be written:

$${}_0^t\mathbf{S} = \frac{\partial W}{\partial\, {}_0^t\mathbf{E}} = \mu\, [\mathbf{I} - I_3^{-0.5(a+2)}\, {}_0^t\mathbf{H}] \tag{7}$$

For a rubber-type elastomer, regarded as a hyperelastic material,

$$ {}_0\mathbf{D} = \frac{\partial^2 W}{\partial {}_0^t\mathbf{E}\, \partial {}_0^t\mathbf{E}} = \frac{d\, {}_0^t\mathbf{S}}{d\, {}_0^t\mathbf{E}} \tag{8} $$

Additionally introducing a tensor of the fourth order

$$ {}_0^t\mathbf{Q} = \frac{\partial\, {}_0^t\mathbf{H}}{\partial\, {}_0^t\mathbf{C}} \tag{9} $$

yields a tensor of incremental constitutive relationship (tangent)

$$ {}_0\mathbf{D} = \mu(a + 2) I_3^{-0.5(a+4)}\, {}_0^t\mathbf{H}\, {}_0^t\mathbf{H} - 2\mu I_3^{-0.5(a+2)}\, {}_0^t\mathbf{Q} \tag{10} $$

Material constants μ and a are determined from the results of experiments performed for each type of rubber. In axisymmetric problems – ${}_0^tC_{31} = {}_0^tC_{32} = 0$ and ${}_0^tC_{33} = 1$.

Material constants matrix (constitutive matrix) is constructed in accordance with the principles of composite material mechanics (JONES 1975).

Node displacements in the finite element axisymmetric tyre model are determined by the incremental method with iterations, based on a matrix equation (BATHE 1982):

$$ ({}_0^t\mathbf{K}_L + {}_0^t\mathbf{K}_{NL})\, \mathbf{u}^{(i)} = {}^{t+\Delta t}\mathbf{R} - {}^{t+\Delta t}_{0}\mathbf{F}^{(i-1)} \tag{11} $$

where:
$\mathbf{u}^{(i)} = {}^{t+\Delta t}\mathbf{u}^{(i)} - {}^{t+\Delta t}\mathbf{u}^{(i-1)}$, ${}^{t+\Delta t}\mathbf{u}^{(0)} = {}^{t}\mathbf{u}$, ${}^{t+\Delta t}_{0}\mathbf{F}^{(0)} = {}_0^t\mathbf{F}$, whereas ${}^{t+\Delta t}\mathbf{R}$ represents a vector of nodal external forces which act on a tyre. The index in brackets denotes the number of iteration.

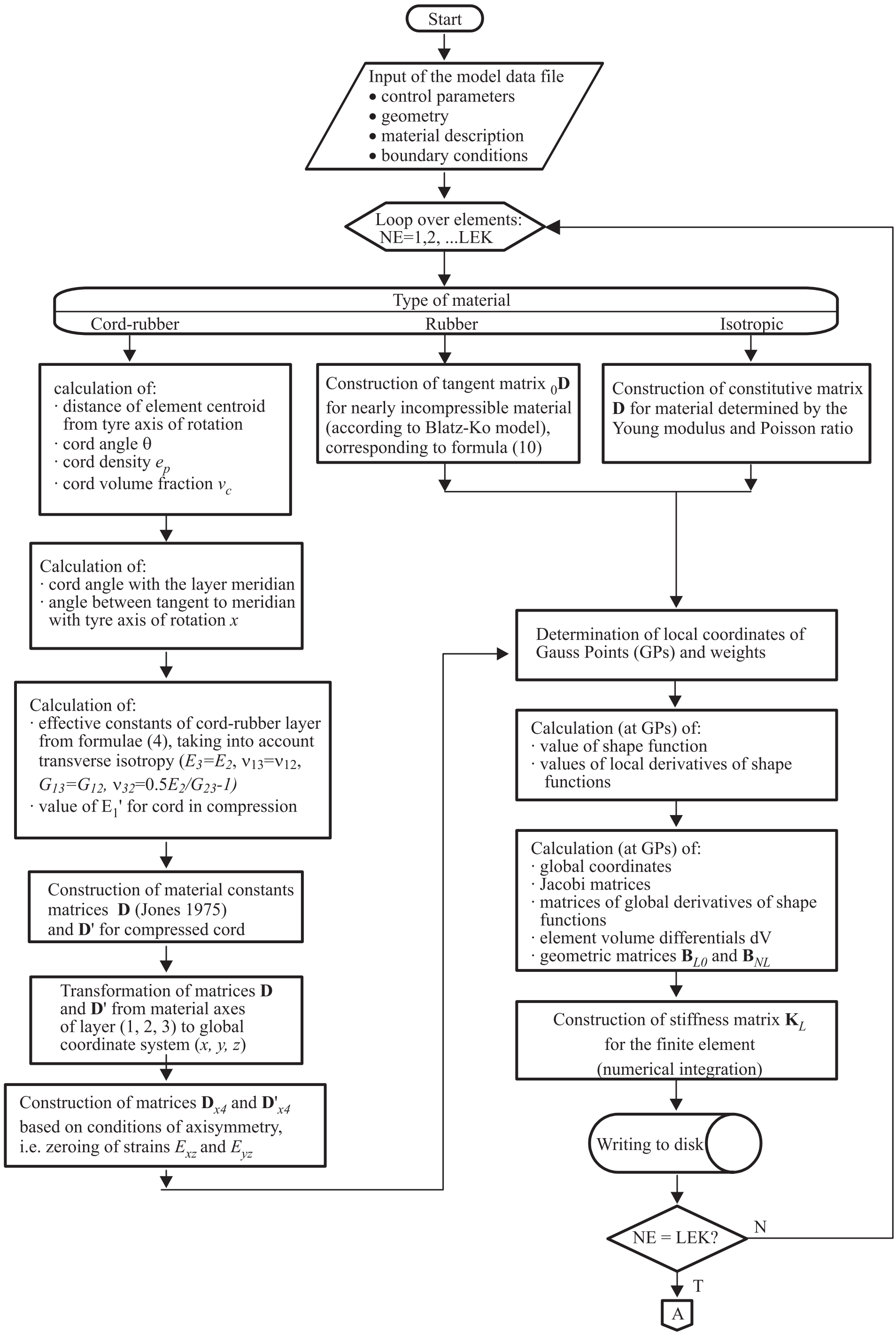
A linear stiffness matrix ${}_0^t\mathbf{K}^L$, non-linear stiffness matrix ${}_0^t\mathbf{K}_{NL}$ and a vector of internal forces ${}_0^t\mathbf{F}$ are calculated from a matrix (linear – ${}_0^t\mathbf{B}_L$ and non-linear – ${}_0^t\mathbf{B}_{NL}$) which occurs in strain-displacement relations, incremental constitutive matrix ${}_0\mathbf{D}$ as well as matrix ${}_0^t\mathbf{S}$ and vector ${}_0^t\hat{\mathbf{S}}$, made up of the components of the second Piola-Kirchhoff stress tensor. It must be stressed that matrix ${}_0^t\mathbf{B}_L$ is the sum of two components: ${}_0^t\mathbf{B}_{L0}$ with constant terms and ${}_0^t\mathbf{B}_{L1}$ with the terms dependent on the level of displacements, *i.e.* which change during the incremental-iterative process.

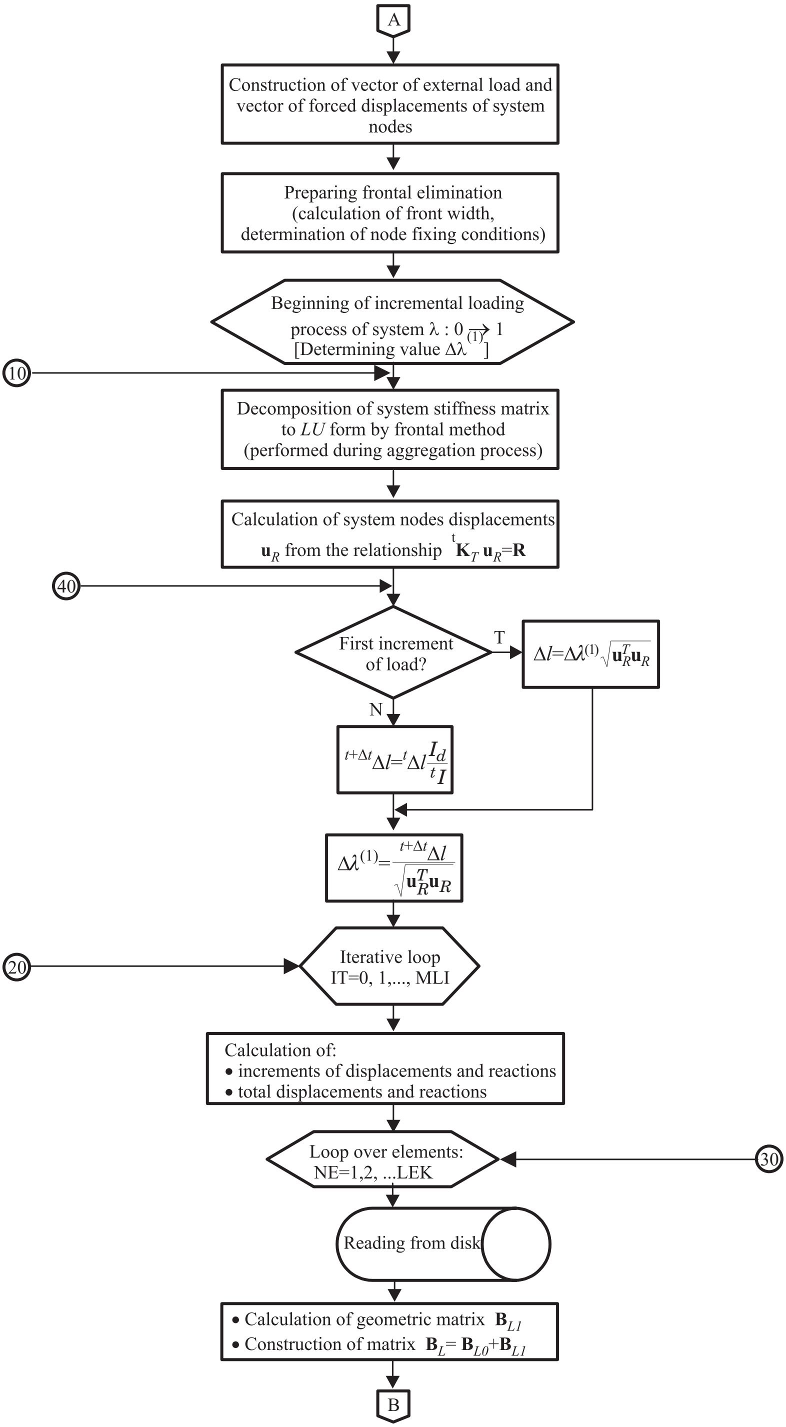
Block diagram of the author's code

The code intended for analysis of axisymmetric models of pneumatic tyres has been developed by the author over many years. Due to the complicated structure of pneumatic tyres, the code is considerably large, with ca. 4000 lines in the FORTRAN language (excluding comment lines). An input data file for the author's code is generated with preprocessing tools, described by PELC (1995b). The data are grouped in sections with specific names (numbers of elements and nodes, node coordinates, types of elements, materials assigned to elements, boundary and load conditions) and have a structure typical for FEM codes, except a significant difference in description of the model material parameters. The data constitute a section in an input data file and contain: name of the material, thickness of the layer (t), cord cross section area (A_c), density of cord in a layer on the drum (e_{p0}), angle of cord to the parallel line of the tyre building drum (θ_0), the layer radius on the drum (r_0), the Young modulus of the cord (E_c), Poisson ratio for the layer rubber or the a parameter for the Blatz-Ko rubber model (ν_r, a), shear modulus for the layer rubber or parameter μ for the Blatz-Ko rubber model (G_r, μ). (*cf.* PELC 2007). Unlike commercial FEM codes, data for the author's code do not contain the values of cord-rubber layer effective constants because for a specific tyre composite they depend on the distance of a layer point from the tyre axis of rotation. The values of those constants are calculated in the relevant subroutine for consecutive elements and material parameter matrices are constructed.

Due to the three selected types of materials in tyre, the code block diagram (Fig. 3) is divided into three paths, with that related to the cord-rubber composite being the most complex. A bimodular characteristic of the cord was assumed with a high value of the Young modulus for stretching and an extremely small value for compression. Therefore, two matrices of material constants are prepared and which of them will be used depends on the state of the current stress in cord, observed during the tyre incremental loading. Due to the assumed rotational symmetry of the problem (i.e. zeroing of deformations E_{xz} and E_{yz}), 4×4 matrices are obtained from full constitutive matrices **D** and **D'**.

An non-inflated tyre has a very low stiffness. Therefore, load increments during the initial phase of simulation must be very small for the iteration process to be convergent. With increasing internal pressure, the system becomes increasingly stiff and load increments may be increased. The author's computation code employs the arc-length control method as presented by CRISFIELD (1981), which makes it possible to automatically match the length of the load step with the degree of non-linearity of the equilibrium path and helps pass through so-called "limit points" (BADUR 1979). The load level in the


Start
Input of the model data file
• control parameters
• geometry
• material description
• boundary conditions
Loop over elements: NE=1,2, ...LEK
Type of material
Cord-rubber
Rubber
Isotropic
calculation of:
· distance of element centroid from tyre axis of rotation
· cord angle θ
· cord density e_p
· cord volume fraction v_c
Construction of tangent matrix ${}_0\mathbf{D}$ for nearly incompressible material (according to Blatz-Ko model), corresponding to formula (10)
Construction of constitutive matrix **D** for material determined by the Young modulus and Poisson ratio
Calculation of:
· cord angle with the layer meridian
· angle between tangent to meridian with tyre axis of rotation x
Calculation of:
· effective constants of cord-rubber layer from formulae (4), taking into account transverse isotropy ($E_3=E_2$, $\nu_{13}=\nu_{12}$, $G_{13}=G_{12}$, $\nu_{32}=0.5E_2/G_{23}-1$)
· value of E_1' for cord in compression
Construction of material constants matrices **D** (Jones 1975) and **D'** for compressed cord
Transformation of matrices **D** and **D'** from material axes of layer (1, 2, 3) to global coordinate system (x, y, z)
Construction of matrices $\mathbf{D}_{x4}$ and $\mathbf{D'}_{x4}$ based on conditions of axisymmetry, i.e. zeroing of strains E_{xz} and E_{yz}
Determination of local coordinates of Gauss Points (GPs) and weights
Calculation (at GPs) of:
· value of shape function
· values of local derivatives of shape functions
Calculation (at GPs) of:
· global coordinates
· Jacobi matrices
· matrices of global derivatives of shape functions
· element volume differentials dV
· geometric matrices $\mathbf{B}_{L0}$ and $\mathbf{B}_{NL}$
Construction of stiffness matrix $\mathbf{K}_L$ for the finite element (numerical integration)
Writing to disk
NE = LEK?
N
T
A


A
Construction of vector of external load and vector of forced displacements of system nodes
Preparing frontal elimination (calculation of front width, determination of node fixing conditions)
Beginning of incremental loading process of system $\lambda : 0 \xrightarrow{(1)} 1$ [Determining value $\Delta\lambda$]
10
Decomposition of system stiffness matrix to *LU* form by frontal method (performed during aggregation process)
Calculation of system nodes displacements $\mathbf{u}_R$ from the relationship ${}^t\mathbf{K}_T\,\mathbf{u}_R=\mathbf{R}$
40
First increment of load?
T
$\Delta l=\Delta\lambda^{(1)}\sqrt{\mathbf{u}_R^T\mathbf{u}_R}$
N
${}^{t+\Delta t}\Delta l={}^t\Delta l\frac{I_d}{{}^tI}$
$\Delta\lambda^{(1)}=\frac{{}^{t+\Delta t}\Delta l}{\sqrt{\mathbf{u}_R^T\mathbf{u}_R}}$
20
Iterative loop IT=0, 1,..., MLI
Calculation of:
• increments of displacements and reactions
• total displacements and reactions
Loop over elements: NE=1,2, ...LEK
30
Reading from disk
• Calculation of geometric matrix $\mathbf{B}_{L1}$
• Construction of matrix $\mathbf{B}_L=\mathbf{B}_{L0}+\mathbf{B}_{L1}$
B

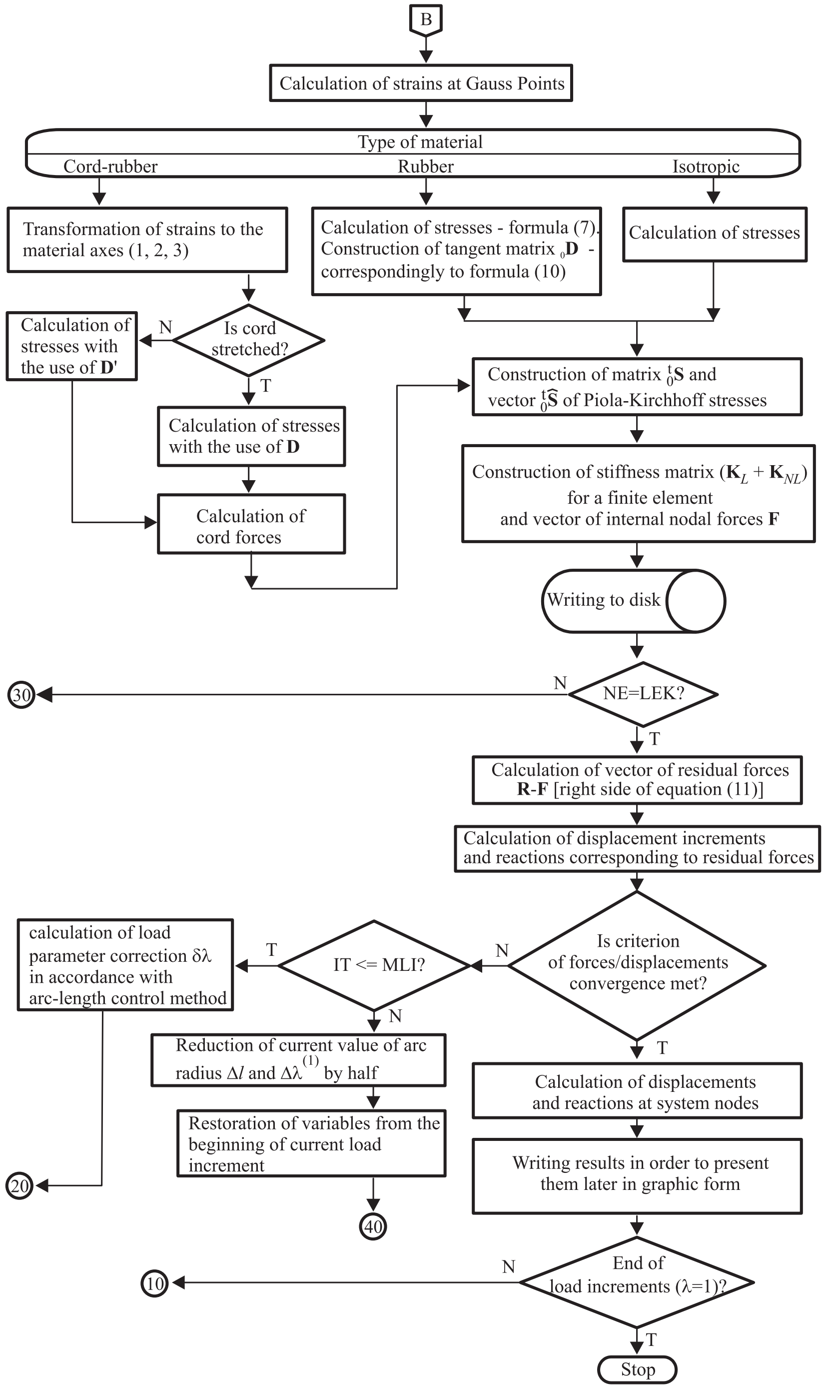


Fig. 3. Block diagram of the code for analysis of axisymmetric deformations of pneumatic tyres

method is controlled by a dimensionless parameter λ, which is equal to zero at the beginning of the incremental process and equal to one at its end. At the beginning of the first load increment, the value of load increment $\Delta\lambda^{(1)}$ is set arbitrarily. The radius of arc Δl, which leads to the path of equilibrium, is calculated with the use of displacement vector of all the nodes of the system ($\mathbf{u}_R$) and the information on the number of iterations performed in the previous increment ^{t}I, whereas I denotes the desired number of iterations. Matrix ${}^t\mathbf{K}_T$ denotes the tangent matrix of the system. It is possible in the subprogram which performs the iteration process to apply two criteria of the process convergence. One of them is based on the Euclidean norm of the nodal residual force vector and the other is related to the iterative displacement increment.

Illustrative calculation results

The author's code can be used to analyse a tyre under axisymmetric loading (internal pressure, centrifugal force of inertia). It is also possible to apply an axisymmetric displacements of a tyre bead nodes. The necessity for axial displacement of the bead nodes towards the tyre's plane of symmetry (y,z) results from a greater spacing of beads in the vulcanisation form than after the tyre has been mounted on a rim. Radial displacements stem from the necessity of the tyre clamping on the rim during the mounting process in order to create the proper friction necessary to transfer the moment of force from the drive and braking and to ensure air-tightness in the contact area between the bead and rim.

By using post-processing tools (PELC 1995b), the author's code can generate the following results in a graphic form:

– a tyre profile before and after deformation,
– element mesh after deformation,
– history of displacement of two characteristic points of a tyre,
– diagrams of force values in cords,
– diagrams of elastic strain energy density in layered materials,
– elastic strain energy distribution in a tyre cross section,
– a map of maximal values of shear strain in a tyre cross section,
– a map of maximal Cauchy stress in sections perpendicular to the tyre profile.

The values presented in diagrams and maps are calculated at the centroids of a finite elements based on the respective values in Gauss points.

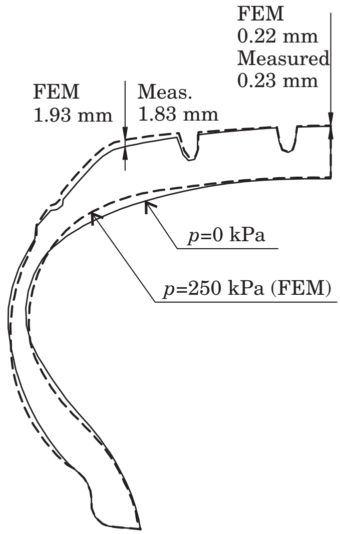


Fig. 4. A passenger car tyre. Initial profile and after inflating to the pressure of 250 kPa

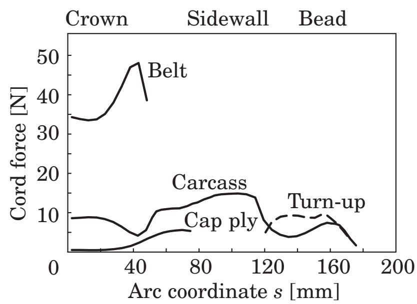


Fig. 5. A passenger car tyre. A diagram of forces in layer cords caused by a pressure of 250 kPa

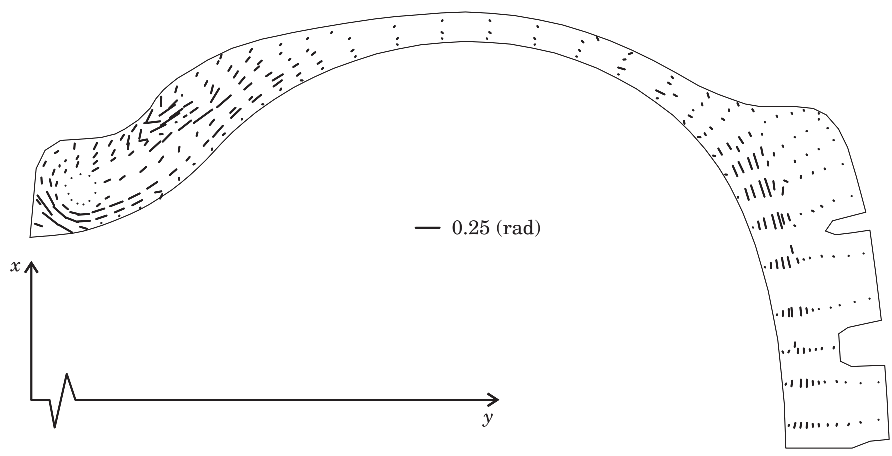


Fig. 6. A truck tyre: a map of extreme values of shear strain in an inflated tyre (p = 800 kPa)

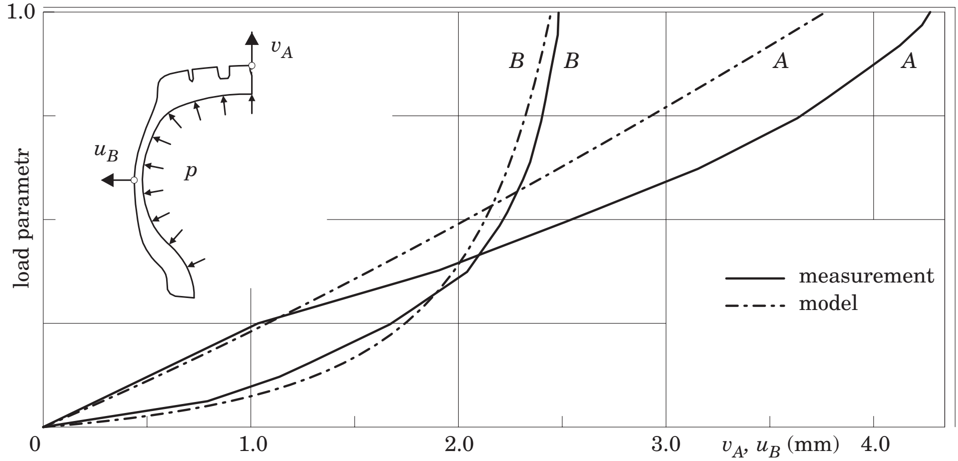


Fig. 7. A truck tyre. A diagram of displacements of characteristic points of a tyre as a function of internal pressure

Summary and conclusions

Pneumatic tyre deformation modelling and strength analysis is a complex issue due to the heterogeneity and non-linear nature of the system. In this case, the method of finite elements seems to be the only tool for reliable modelling of the behaviour of such a system. Due to the heterogeneous structure of a tyre, the block diagram of a code for an axisymmetric tyre model analysis has a complex structure resulting from the distinguishing of three types of materials: cord-rubber composite, nearly incompressible rubber and steel. The geometric non-linearity of a tyre and physical non-linearity of rubber requires that the stiffness matrix of the system should be updated during the process of loading; the incremental-iterative procedure, visible in the block diagram, is linked with it. The author's computer code enables analysis of both passenger tyres and truck tyres.

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References

BADUR J. 1979. *Efektywne wyznaczanie punktów granicznych w nieliniowej mechanice konstrukcji*. IV Konf. Metody Komputerowe w Mechanice Konstrukcji, tom II. Koszalin, kwiecień, s. 203–209.

BATHE K.J. 1982. *Finite element procedures in engineering analysis*. Prentice-Hall, Englewood Cliffs, N.Y.

HALPIN J.C., TSAI S.W. 1969. *Effects of environmental factors on composite materials*. AFML-TR, pp. 67–423.

HOFFERBERTH W. 1956. *Zur Festigkeit des Luftreifens*. Kautschuk u. Gummi. 9: 225–231.

JONES R.M. 1975. *Mechanics of composite materials*. McGraw-Hill London

PELC J. 1992. *Large displacements in tire inflation problem*. Engineering Transactions, 40(1): 103–113.

PELC J. 1995a. *Pomiary i analiza metodą elementów skończonych opony całostalowej do samochodów ciężarowych podczas pompowania*. Acta Acad. Agricult. Tech. Olst. Aedificatio et Mechanica, 26: 15–27.

PELC J. 1995b *Pre-Processing and Post-Processing in Pneumatic Tire Analysis by the Finite Element Method*. Acta Acad. Agricult. Tech. Olst. Aedificatio et Mechanica, 26: 29–36.

PELC J. 1995c. *Orthotropic material model in pneumatic tire analysis*. Proceedings of the XII Polish Conference on Computer Methods in Mechanics, Warsaw–Zegrze, Poland, 9–13 May 1995, pp. 271–272.

PELC J. 2000. *Material Modelling in Cord-Rubber Structures*. Kautsch., Gummi Kunstst. 53(10): 561–565.

PELC J. 2002. *Static three-dimensional modelling of pneumatic tyres using the technique of element overlaying*. Proc. Instn Mech. Engrs, Part D: J. Automobile Engineering, 216(9): 709–716.

PELC J. 2007. *Modelowanie skończonych deformacji opon pneumatycznych,* Rozprawy i Monografie 134, Wyd. UWM.