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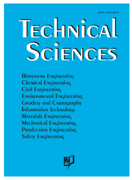

ANALYSIS OF NATURAL FREQUENCY OF FLEXURAL VIBRATIONS OF A SINGLE-SPAN BEAM WITH THE CONSIDERATION OF TIMOSHENKO EFFECT

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Abstract

This paper presents general solution of boundary value problem for constant cross-section Timoshenko beams with four typical boundary conditions. The authors have taken into consideration rotational inertia and shear strain by using the theory of influence by Cauchy function and characteristic series. The boundary value problem of transverse vibration has been formulated and solved. The characteristic equations considering the exact bending theory have been obtained for four cases: the clamped boundary conditions; a simply supported beam and clamped on the other side; a simply supported beam; a cantilever beam. The obtained estimators of fundamental natural frequency take into account mass and elastic characteristics of beams and Timoshenko effect. The results of calculations prove high convergence of the estimators to the exact values which were calculated by Timoshenko who used Bessel functions. Characteristic series having an alternating sign power series show good convergence. As it is shown in the paper, the error lower than 5% was obtained after taking into account only two first significant terms of the series. It was proved that neglecting the Timoshenko effect in case of short beams of rectangular section with the ratio of their length to their height equal 6 leads to the errors of calculated natural frequency: 5%÷12%.

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Introduction

Mechanical vibrations, in particular flexural vibrations are common in load-bearing structures, so designers have to perform calculations for dynamics to protect the structure from fatigue damage. Therefore, the dynamic calculations are important subject of interest in both engineering theory and practice. The initial stage of the calculation is to solve the boundary problem as results from vibrations which are determined by the natural frequencies and the corresponding mode shapes (TIMOSHENKO et al. 1985). In most cases, long an thin beams alone may be used to derive the equation of vibration by a simplified theory of Euler-Bernoulli. However, the experience shows that simplified theory can be applied for slender beams if one deals with higher vibration frequencies calculated. The development of industry and the construction of equipment particularly exposed to dynamic loads have caused the need to refine the technical calculation methods used in the classical strength of materials. In 1914, H. Lamb pointed out that even in the simplest case of an impact loaded beam, the elementary vibration equation given by Euler and Bernoulli is not true. This equation leads to a result showing that the impact of the suddenly applied load propagates at an infinite speed. In order to eliminate this error, the strength methods were abandoned and, as proposed by Timoshenko, corrections due to the Timoshenko effect were introduced TIMOSHENKO 1971). They took into account the effect of transverse forces on bending and rotational inertia (Rayleigh correction). Table 1 shows the related exemplary results obtained by (SOLECKI, SZYMKIEWICZ 1964)

Table 1

The values of the first three frequencies for the above data in the case of a simply supported beam, with the consideration of Timoshenko effect

Frequency number n	$\tilde{\omega}$ [rad/s] (without effect)	$\tilde{\omega}$ [rad/s] (with effect)	Difference [%]
1	6	5.73	5
2	24	20.3	15
3	54	39.4	27

Although Timoshenko beam theory is much more complex than the fundamental one, it is much simpler than the solution of the three-dimensional theory of elasticity. The introduced correction caused a significant qualitative change in the beam vibration equation. It was possible, however, to capture the experimentally confirmed wave nature of the phenomenon, and with sufficient efficiency, it was also possible to discuss the results originating from the area of the wave fronts.

Figure 1 presents the dependence, drawn with the broken line, which was obtained by the Bernoulli theory, or with the dotted line, as stated by the refined theory. The continuous lines show the dependencies resulting from the exact solution of dynamic Lame equations for a bar with the circular cross-section. The curves corresponding to the first three modes of vibrations were marked with digits I, II, III, respectively. The correlation of the curves shown in Figure 1 shows that the refined theory can provide satisfactory approximations for the lower orders of vibration. For large values, the *k*-divergence is rather important.

If wave number $k \to \infty$ the exact value for velocity of surface Raleigh waves is c_R, whereas in Timoshenko theory the limit value is the velocity of longitudinal waves c_0.

For each fixed wave number, there are two phase speeds c_1, c_2 which mainly correspond to the bending and cutting forms of the propagating waves. At $k_2R_2 \ll 1$; $c_1 \approx kRc_0$; $c_2 \approx \frac{c_t}{kR}$ hence c_1 occurs with the velocity of bending waves, calculated on the base of the fundamental theory, with c_t being the velocity of deformation. The velocity c_2 at $kR \to 0$ increases to infinity (BOLOTIN 1979).

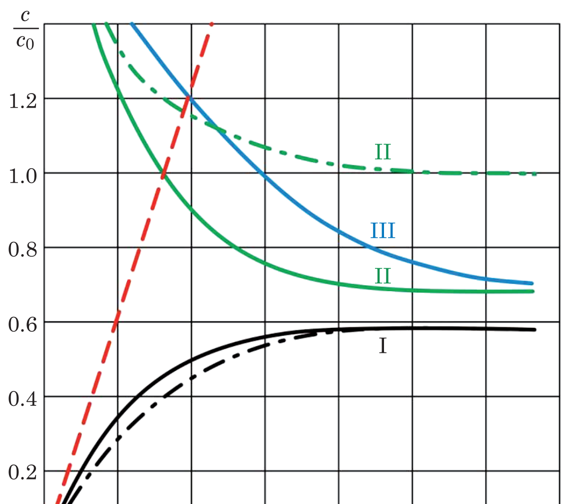


Fig. 1. Dependency of phase velocity of bending waves in rods on propagation coefficient, obtained by the use of a number of theories: $c = k(EJ/\rho F)^{1/2}$, $R = (J/F)^{1/2}$, J, F – moment of inertia and the cross section area of the beam, ρ – denotes density of a material, E – Young modulus of elasticity λ – length of wave
Source: based on BOLOTIN (1979).

In the current work we solved the boundary problem of vibration bent for the four boundary conditions: both ends clamped, a simply supported end and the other clamped, soft simply supported on one end and hard simply supported on the other end, and the cantilever. To solve those problems, the Cauchy functions and series of characteristics developed under the personal guidance of prof. Zoryj were applied, for example in work (JAROSZEWICZ et al. 2004). Exact formulas for the coefficients of a number of characteristic were derived. Moreover, taking into account only the first two significant terms of the series resulted in error not exceeding 5%. On the basis of the derived formulas, numerical values of the coefficients of the short series were calculated for a reinforced beam with a rectangular profile of the ratio of length to height equal to 6. Disregarding the Timoshenko effect leads to the considerable calculation error. In order to calculate the fundamental frequency and the next two (the second and the third ones), double Bernstein estimates were used which appeared to be consistent with the exact values for the cantilever beam. The problem of impact of Timoshenko effect on vibrations of complex dynamic systems has been widely covered in literature, particularly in works (WU, CHIANG 2004, MAMANDI, KARGARNOVIN 2011, MOEENFARD et al. 2011, CHEN 2014, ZHANG et al. 2014, HSU 2016). Particular attention should be paid to publications (CAZZANI et al. 2016a, 2016b), in which the Timoshenko effect is evaluated for systems with variable mass and elastic parameters.

To solve the boundary problem, we use the methods of the Cauchy function and the method of characteristic series as an extension of Bernstejn's spectral functions and double (lower and upper) estimators Bernstejn-Kieropian, which were developed in 1977-2000 at the Lviv Polytechnic by prof. L. Zoryj and at the Bialystok University of Technology by prof. J. Jaroszewicz. It was shown that the above methods are effective for solving linear 4^{th} order differential equations with fixed coefficients and variable coefficients containing singularities. Thanks to this method, the coupling conditions can be omitted, which increase the number of boundary conditions and increase the degree of the characteristic determinant.

The forms of characteristic equations proposed in the work allow to write the functional relationships between the frequency of the vibration form and the mass-elastic characteristics of the models. This allows not only to calculate the frequency and form of vibrations, but also to optimize and influence the formation of specific dynamic behaviors. For the construction of solutions of analogous differential equations, Bessel special functions were commonly used, which require solving complex systems of algebraic equations. Therefore, the proposed solution structure using the Cauchy function is much more effective and the characteristic equations are derived in the form of fast convergent power series of the frequency parameter. Double estimators give limits in which the exact frequency value is located using elementary functions. By deriving a greater number of linear terms in the series, two or more frequencies can be determined without the need for numerically solving high order characteristic equations.

The equation of flexural vibrations

In the case of load-bearing structures, like beams, in which height of cross section is larger than the span or length of the beam, and when higher natural frequencies are calculated, it is necessary to consider the influence of shear strain with regard to rotational inertia and shear, in literature known as the effect of Timoshenko (TIMOSHENKO 1971, THOMAS, ABBAS 1975).

Not considering these effects leads to the significant divergence with respect to calculations based on the Timoshenko theory (TIMOSHENKO et al. 1985, SOLECKI, SZYMKIEWICZ 1964).

For short beams, when calculating higher frequencies, when the wavelength of the deformation is comparable to the transverse beam dimensions, the technical theory of bending vibration, which does not take into account the shear and inertia of the rotation of the cross sections, results in large errors in calculations i.e. overestimates frequencies. The beam shape for the first mode of vibration is shown in Figure 2.

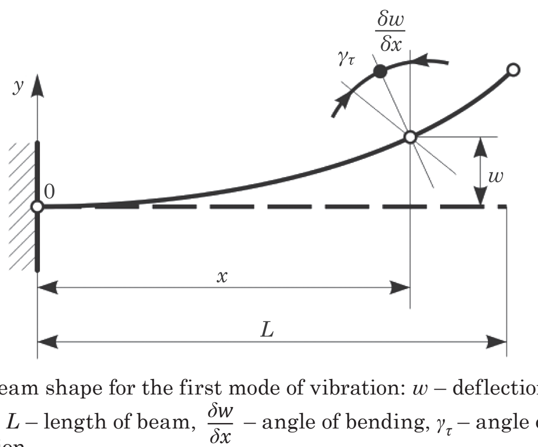


Fig. 2. The beam shape for the first mode of vibration: w – deflection of a beam, x – length of axis, L – length of beam, $\frac{\delta w}{\delta x}$ – angle of bending, γ_τ – angle of a cross-section
Source: own elaboration.

Timoshenko beam is a sufficiently generalized calculation model for the studied load-bearing structure, with the consideration of influence of rotational angle and shear strain. The loading state of the element section of the beam is presented in Figure 3 and 4.

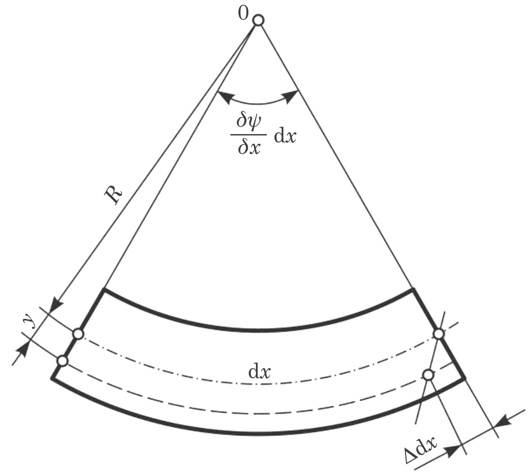


Fig. 3. Schematic diagram of the internal loading of the beam element resulting from the rotational inertia and inextensional strain: R – radius of element, y – length from neutral axis to section axis, dx – length of elementary element, Δdx – elongation, ψ – angle, $\frac{\delta\psi}{\delta x}$ dx– elastic deformation
Source: based on VASYLENKO, ALEKSEJIUK (2004).

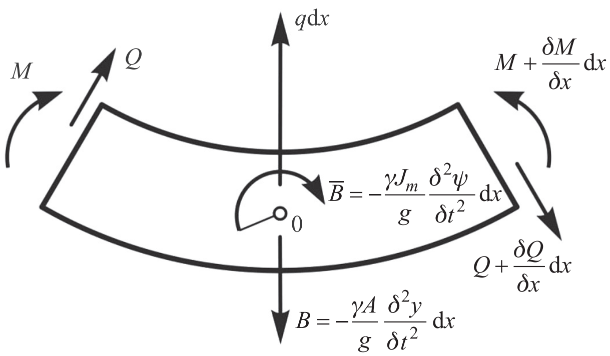


Fig. 4. Schematic diagram of loading of beam elements with respect to rotational inertia and non-dilatational strain: A – cross sectional area, M – moment of element bending, Q – shear power, B –inertia force, J_m – moment of inertia, g – acceleration of gravity, γ – mass density
Source: based on VASYLENKO, ALEKSEJIUK (2004).

For simplicity we assume that the beam performs the first bending mode of vibration for which (see: VASYLENKO, ALEKSEJIUK 2004):

$$\Theta = \frac{\partial w}{\partial x} + \gamma_\tau \tag{1}$$

Then, on the basis of Figure 2, we obtain:

$$Q = -KGF\left(\frac{\partial w}{\partial x} - \Theta\right) \tag{2}$$

where:

$K = \frac{Jb}{FS}$ – the coefficient applied to establish shear deformation,

G – Kirchhoff module of elasticity,

J – a moment of cross section inertia,

F – cross section plane,

S – a moment of inertia of a part of the section located at one side of the axis of symmetry,

b – width of cross section in the neutral plane.

Taking into account the forces acting on the dx element, d'Alembert's inertial forces from the linear w and angular Θ displacements of the sections (Fig. 2), we apply equations of dynamic equilibrium $\sum Y = 0$, $\sum M_0 = 0$. From these equations, rejecting infinitely small quantities of the second order, we obtain the following relations:

$$\frac{\partial Q}{\partial x} = q - \rho F \frac{\partial^2 w}{\partial t^2},\ \frac{\partial M}{\partial x} = Q + J_m \frac{\partial^2 \Theta}{\partial t^2} \tag{3}$$

Substituting equation (2) to (3), and taking into account known relations (Vasylenko, Aleksejiuk 2004), we obtain the equations of forced vibrations of the elastic Timoshenko beam:

$$\begin{cases} \rho F \frac{\partial^2 w}{\partial t^2} - KGF\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \Theta}{\partial x}\right) = q(x,t) \\ EJ \frac{\partial^2 \Theta}{\partial x^2} + KGF\left(\frac{\partial w}{\partial x} - \Theta\right) - J_m \frac{\partial^2 \Theta}{\partial t^2} = 0 \end{cases} \tag{4}$$

This work examines the method of analysis of transverse vibrations of the given beam with the use of Cauchy function and characteristic series (Jaroszewicz, Zoryj 2000). Earlier, Jaroszewicz applied Cauchy function to analyze both static and dynamic problems of the beam systems (Jaroszewicz 1999, Jaroszewicz et al. 2014).

Formulation of the boundary problem

From the equation (4), for a constant cross-section beam EJ=const., GJ=const., ρF=const with $q(x,t) = 0$, we obtain the equation of motion of transverse free vibrations of load-bearing beam system, the elements of which are shown in Figure 1. This equation can be written (Jaroszewicz et al. 2004) in the form:

$$\rho F \frac{\partial^2 w}{\partial t^2} + EJ \frac{\partial^4 w}{\partial x^4} - \rho J \left(1 + \frac{E}{KG}\right) \frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\rho^2 J}{KG} \frac{\partial^4 w}{\partial t^4} = 0 \tag{5}$$

where:

ρ – denotes density of a material,
E – Young modulus of elasticity,
w – a deflection of the beam,
x – the longitudinal beam coordinate,
t – time.

Determination of free frequencies needs solving the appropriate boundary value problem. Substitution of $w = y\left(\frac{x}{l}\right)e^{\lambda t}$ with (5) and separation of two variables x and t give the differential equation with respect to x:

$$y^{\mathrm{IV}} - \beta y'' - \delta y = 0, \; (0 \leq x \leq 1) \tag{6}$$

where:

$\beta = \lambda^2 \rho J (1 + \frac{E}{KG}) l^2 (EJ)^{-1} = \lambda^2 A$,
$\delta = -(EJ)^{-1}(\rho F \lambda^2 + \rho^2 \frac{J}{KG} \lambda^4) l^4 = -(\lambda^2 B + \lambda^4 C)$,
$w^2 = -\lambda^2$ – the second power of frequency.

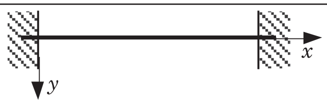
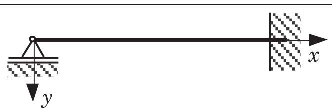
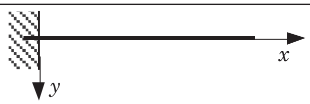
The formulas above contain the following symbols:

$$A = \frac{\rho l^2}{E}(1 + \frac{E}{KG})\,;\; B = \frac{\rho F l^4}{EJ}\,;\; C = \frac{\rho^2 l^4}{EKG} \tag{7}$$

Four cases of boundary conditions (Tab. 2), corresponding to the most frequently occurring supports of beams were analyzed of (JAROSZEWICZ et al. 2004).

Table 2

Four cases of boundary conditions

	$y(0) = y'(0) = 0$	$y(1) = y'(1) = 0$	(8)
	$y(0) = y''(0) = 0$	$y(1) = y'(1) = 0$	(9)
	$y(0) = y''(0) = 0$	$y(1) = y''(1) = 0$	(10)
	$y(0) = y'(0) = 0$	$y''(1) = y'''(1) = 0$	(11)

The general solution

The general solution of equation (6) can be expressed in the following form (Jaroszewicz, Zoryj 2000):

$$y = C_0\varphi + C_1\varphi' + C_2\varphi'' + C_3\varphi''' \tag{12}$$

in which the function $\varphi(x, \beta, \delta)$ and its x derivatives are defined below, and where C_0, C_1, C_2, C_3 denote arbitrary constants.

The Cauchy function appearing in the equation (12) is described with the formulas:

$$\varphi(x,\beta,\delta) = \sum_{K=0}^{\infty} \frac{J_K x^{K+3}}{(K+3)!} = \frac{1}{\mu^2+\nu^2}\left(\frac{\mathrm{sh}\mu x}{\mu} - \frac{\sin\nu x}{\nu}\right) \tag{13}$$

$$J_0 = 1,\ \ J_K = \beta J_{K-2} + \delta J_{K-4},\ \ J_j = 0 \quad \text{for } j < 0$$

$$\mu^2 = \sqrt{\frac{1}{4}\beta^2 + \delta} + \frac{1}{2}\beta,\ \nu^2 = \sqrt{\frac{1}{4}\beta^2 + \delta} - \frac{1}{2}\beta$$

Characteristic equations corresponding with the first three types of boundary conditions

Substitution of the form (12) in the boundary conditions (8) from Table 2, corresponding to the clamped-clamped beam, results in the following characteristic equation:

$$F = F(1,\beta,\delta) = ((\varphi')^2 - \varphi\varphi''))_{x=1} = 0 \tag{14}$$

where:

$$F = \sum_{n=0}^{\infty} (-1)^n 2^{2n+1} f_n(x,\beta)\delta^n \tag{15}$$

$$f(x,\beta) = \sum_{m=0}^{\infty} C_{2n+1+m}^{2n+1} \frac{\beta^m x^{4n+2m+4}}{(4n+2m+4)!} \tag{16}$$

C_{2n+1+m}^{2n+1} – binominal coefficient.

Subsequent substitution of the general solution (12) in the conditions (9) yields the characteristic equations of a similar structure as before, on whose left side there is the following derivative of the function F from (15):

$$F'(1,\beta,\delta) = 0 \tag{17}$$

Subsequently, substitution of (12) in (10) leads to the following derivative on the left-hand side of (10):

$$F''(1,\beta,\delta)=0 \tag{18}$$

Now we can construct function for boundary conditions (11) in power series form with respect to the coefficient λ^2. For this purpose, a few first functions can be calculated from (16):

$$\begin{aligned}
f_0(x,\beta)&=\frac{x^4}{4!}+\frac{2x^6\beta}{6!}+\frac{3x^8\beta^2}{8!}+\ldots\\
f_1(x,\beta)&=\frac{x^8}{8!}+\frac{4x^{10}\beta}{10!}+\frac{10x^{12}\beta^2}{12!}+\ldots\\
f_2(x,\beta)&=\frac{x^{12}}{12!}+\frac{6x^{14}\beta}{14!}+\frac{21x^{16}\beta^2}{16!}+\ldots\\
f_3(x,\beta)&=\frac{x^{16}}{16!}+\frac{8x^{18}\beta}{18!}+\frac{36x^{20}\beta^2}{20!}+\ldots\\
f_4(x,\beta)&=\frac{x^{20}}{20!}+\frac{10x^{22}\beta}{22!}+\ldots
\end{aligned} \tag{19}$$

and a few first terms of the series (15) are:

$$F=2f_0-2^3f_1\delta+2^5f_2\delta^2-2^7f_3\delta^3+\ldots \tag{20}$$

The right-hand side of the relation (20) can be written in a form of a series:

$$\Delta \equiv a_0+a_1\lambda^2+a_2\lambda^4+a_3\lambda^6+\ldots \quad |_{x=1}=0 \tag{21}$$

presented as a characteristic series, also named the Bernstein spectral function of the formulated boundary value problem. The characteristic equation (21) in the form of the power series in relation to the frequency parameter is obtained by substituting the general form of equation (12) to the boundary conditions from Table 2, equating equations of the system of four algebraic equations to zeros, and then using the conditions for non-zero values for integration constants C_0, C_1, C_2, C_3.

Formulas for the first four terms of a series (16) for the example presented in Table 2 have been determined by using relations (17)÷(20):

$$a_0 = \frac{2}{4!}, \quad a_1 = \frac{4}{6!}A + \frac{2^3}{8!}B \tag{22}$$

$$a_2 = \frac{3 \cdot 2}{8!}A^2 + \frac{2^3 \cdot 4}{10!}AB + \frac{2^3}{8!}C + \frac{2^5}{12!}B^2$$

$$a_3 = \frac{4 \cdot 2}{10!}A^3 + \frac{2^3 \cdot 10}{12!}A^2B + \frac{2^3 \cdot 4}{10!}AC + \frac{2^5 \cdot 6}{14!}AB^2 + \frac{2^6}{12!}BC + \frac{2^7}{16!}B^3$$

$$a_4 = \frac{5 \cdot 2}{12!}A^4 + \frac{2^3 \cdot 20}{14!}A^3B + \frac{2^3 \cdot 10}{12!}A^2C + \frac{2^5 \cdot 21}{16!}A^2B^2 + \frac{2^6 \cdot 6}{14!}ABC + \ldots$$

where A, B, C are constants defined with (7).

The above formulas (22), correspond to a clamped-clamped support – the case (8). For the cases (9) and (10), the formulas for subsequent coefficients of the characteristic series (21) can be written, bearing in mind that $F'(1, \beta, \delta) = 0$ in the case (9) and $F''(1, \beta, \delta) = 0$ in the case (10).

Having obtained coefficients (22), Berstein-Keropian double (upper and lower) estimators have been applied to calculate base frequency and the next two (the second and the third ones). Results of the calculation are presented below, in section 6. It should be noted here that the first and second frequencies have been received with a great accuracy, and the third and fourth ones are determined with approximate values. Approximate values may be enhanced by means of iteration method with use of formula (12) and the exact expression for the function (15):

$$F = \frac{1}{(\mu^2 + \nu^2)^2}\left[(\mu^2 + \nu^2)\frac{\mathrm{sh}\mu \sin \nu x}{\mu\nu} + 2(1 - \mathrm{ch}\mu x \cos \nu x)\right] \tag{23}$$

Characteristic equations in the case of cantilever beam boundary conditions

In case of cantilever beam with Timoshenko effect taken into account, for which boundary conditions are in the form (11) (Tab. 2), the characteristic equation can be obtained in the following form:

$$1 - \delta F(1, \beta, \delta) = 0 \tag{24}$$

which, by substituting δ in (24) and accounting for (7), reads:

$$1 + (\lambda^2 B + \lambda^4 C) \cdot F(1, \beta, \delta) = 0 \tag{25}$$

Then, consideration of (12) leads to the following coefficients of characteristic series:

$$a_0 = 1;\ a_1 = \frac{2}{4!}B;\ a_2 = \frac{2}{8!}B^2 + \frac{4}{6}AB + C$$

$$a_3 = \frac{2^5}{12!}B^3 + \frac{3\cdot 2}{8!}A^2B + \frac{2^3\cdot 4}{10!}AB^2 + \frac{2^3}{8!}CB + \frac{4}{6!}AC + \frac{2^3}{8!}BC \tag{26}$$

$$a_4 = \frac{2^7}{16!}B^4 + \frac{4\cdot 2}{10!}A^3B + \frac{2^3\cdot 10}{12!}A^2B^2 + \frac{2^6}{10!}ABC + \frac{2^5\cdot 6}{14!}AB^3 + \frac{2^5\cdot 3}{12!}B^2C +$$
$$+ \frac{3\cdot 2}{8!}A^2C + \frac{2^3}{8!}C^2 \ldots..$$

The final form of equation (25) is received by means of transformation of the expressions (26)

$$1 - \delta F(x, \lambda) = 0 \tag{27}$$

where:

$$F(x,\lambda) = \sum_{n=0}^{\infty}\lambda^{2n}\left\{\sum_{j=0}^{n/2}C^j\sum_{m=j}^{n-j}2^{2m+1}\frac{x^{2n+2m+2j+4}}{(2n+2m+2j+4)!}C_{m+n-j+1}^{2m+1}C_m^jA^{n-m-j}B^{m-j}\right\} \tag{28}$$

The result in the form corresponding to the case without Timoshenko effect and values of estimators is not difficult to obtain from formulas (27):

$$a_0 = 1;\ a_1 = \frac{2}{4!}B;\ a_2 = \frac{2^3}{8!}B^2 \tag{29}$$

$$\sqrt{\frac{EJ}{ml^4}}1.8749 < \omega_1 < 1.8751\sqrt{\frac{EJ}{ml^4}} \tag{30}$$

Values of coefficients of base frequencies which were calculated on base of Bernstein estimators [1] are in agreement with the exact values calculated by means of Krylov-Prager (THOMAS, ABBAS 1975) function for cantilever.

Calculation results of base frequency for four cases of supports, according to the proposed method

Table 3, presented below, contains formulas for the first three terms of a series without Timoshenko effect.

The application of Cauchy influence function and characteristic series to the analysis of the transverse vibration of short beams gave results in the form of characteristic power series (21) for which closed formulas for the first four terms of series (22) and (26) can be formulated. The calculations of frequency

Table 3

Final formulas for the first three terms of a series without Timoshenko effect

Boundary conditions from Table 2	a_0	a_1	a_2
(8)	$\frac{2}{4!}$	$\frac{2^3}{8!}B$	$\frac{2^5}{12!}B^2$
(9)	$\frac{2}{3!}$	$\frac{2^3}{7!}B$	$\frac{2^5}{11!}B^2$
(10)	$\frac{2}{2!}$	$\frac{2^3}{6!}B$	$\frac{2^5}{10!}B^2$
(11)	1	$\frac{2}{4!}B$	$\frac{2^3}{8!}B^2$

were adjusted for the first three terms of the series. On the basis of the simplest Bernstein estimators, and by the use of previously listed formulas, the base frequency can be calculated (BERNSTEIN, KIEROPIAN 1960):

$$\frac{a_0}{\sqrt{a_1^2 - 2a_0a_2}} < \omega_1^2 < \frac{2a_0}{a_1 + \sqrt{a_1^2 - 4a_0a_2}} \tag{31}$$

If one knows a_3 and a_4 and the coefficients (22) and (26) for the frequency of equation (21), the plate first three frequencies can be estimated and the approximate value of the fourth frequency can be determined. First, let us calculate the following numbers:

$$B_1 = \frac{a_1}{a_0} = \frac{1}{60}, \quad B_2 = \left(\frac{a_1}{a_0}\right)^2 - 2\frac{a_1a_2}{a_0^2} = 0.0001786 \tag{32}$$

and the ratio:

$$\frac{B_2}{B_1^2} = 0.64288 \tag{33}$$

for which the corresponding results are given in BERNSTEIN, KIEROPIAN (1960, Tab. 21, p. 236). The following formulas are used:

$$\omega_i = \gamma_i \frac{1}{R^2}\sqrt{\frac{D_0}{\rho h_0}}, \quad (i = 1, 2, 3) \tag{34}$$

$$(\gamma_i)_- = \sqrt[4]{\frac{\varphi_i}{B_i}}, \quad (\gamma_i)_+ = \sqrt[4]{\frac{\beta_i}{B_1}}, \quad (i = 1, 2, 3) \tag{35}$$

$$\gamma_4 \approx \sqrt[4]{\frac{\psi}{B_1}} \tag{36}$$

Expression (36) defines the approximate estimate of the fourth eigenvalue parameter according to Bernstejn-Kieropian tables. Having the first 3 coefficients of equation (21) a_0, a_1, a_2 calculated, one can determine the estimators of the first three vibration frequencies and approximate the fourth one. Bernstejn and Kieropian developed formulas for estimators of higher frequencies and placed them in tables of relations between them. When calculating the ratio (33), we read on the basis of [1] the values: φ_1, φ_2, φ_3, β_1, β_2, β_3 and ψ and then we calculate the upper and lower estimators of the higher frequencies.

Taking relations (32)–(36), and substituting $a_0 = 1$, and a_1 and a_2 from (22) into these relations, we obtain:

$$
\begin{gathered}
8.64 < \gamma_1 < 8.85 \\
10.06 < \gamma_2 < 20.08 \\
31.61 < \gamma_3 < 52.56 \\
\gamma_4 \approx 66.61
\end{gathered}
\tag{37}
$$

Coefficients of the characteristic series of the present beam without Timoshenko effect, manifested by zero values A and C, are presented in Table 3. To calculate the subsequent frequencies, Bernstein tables must be applied (BERNSTEIN, KIEROPIAN 1960).

The calculation results of the lower and upper estimators of the base frequency, without Timoshenko effect, conducted by using symbols from Table 3, correspond to the exact values (TIMOSHENKO 1971, THOMAS, ABBAS 1975, SOLECKI, SZYMKIEWICZ 1964) and they are listed in Table 4.

The results of calculation of the lower and upper estimators of the base frequency are obtained with Timoshenko effect for concrete beam with a rectangular cross-section, in which: the ratio of length to cross-sectional height is:

Table 4

Values of estimators for the base frequency in the case of four support without Timoshenko effect

Boundary conditions	Low estimator $\frac{1}{\pi}\sqrt{\frac{\rho F l^4}{EJ}}(\omega_1)_-$	High estimator $\frac{1}{\pi}\sqrt{\frac{\rho F l^4}{EJ}}(\omega_1)_+$	Exact value $\frac{1}{\pi}\sqrt{\frac{\rho F l^4}{EJ}}(\omega_1)$
(8)	4.720	4.732	4.730
(9)	3.818	3.924	3.927
(10)	3.140	3.143	π
(11)	1.802	1.875	1.870

l/h = 6, Young modulus and Kirchoff modulus relationship reads $E/G = 2.5$, and shape of the section is $K = Jb/FS = 2/3$, the area of cross-section equals: $F = b \cdot h = 0.5$ m^2, the moment of inertia of a cross-section is: $J_x = bh^2/12 = 0.042$ m^4, S – the first moment of the section area located on one side of the axis neutral to this axis, ρ = 24,000 Ns^2/m – the density of the material. Parameters A, B, C, calculated with formulas (7) are as follows: A = 0.029882 1/s^2, B = 2.718 1/s^2, C = 0.000148 1/s^4. These results for geometrical and material exemplary parameters are listed in Table 5. They correspond to the calculation results presented in the work (SOLECKI, SZYMKIEWICZ 1964).

Table 5

Values of the first three coefficients of the characteristic series for the base frequency for the cases from Table 2, without Timoshenko effect.

Boundary conditions	a_0	$a_1 \cdot 10^{-4}$	$a_2 \cdot 10^{-6}$	$\tilde{\omega}_1$ [rad/s]	$(\omega_1)_+$ [rad/s]	$(\omega_1)_-$ [rad/s]	$(\omega_1)_{sr}$ [rad/s]
(8)	1/12	5.4	0.5	12.43	13.65	13.51	13.58
(9)	1/3	43.14	5.9	8.79	8.96	8.87	8.91
(10)	1	302	6.5	5.99	5.58	5.56	5.57
(11)	1	2264	1465	2.11	2.13	2.13	2.13

Table 6

Values of the first three coefficients of the characteristic series for the base frequency for the cases from Table 2, with Timoshenko effect

Boundary conditions	a_0	$a_1 \cdot 10^{-4}$	$a_2 \cdot 10^{-6}$	$\tilde{\omega}_1$ [rad/s]	$(\omega_1)_+$ [rad/s]	$(\omega_1)_-$ [rad/s]	$(\omega_1)_{sr}$ [rad/s]
(8)	1/12	7.05	1.41	11.87	13.57	12.68	13.13
(9)	1/3	57.01	6.3	8.16	8.25	8.22	8.24
(10)	1	456	7.03	5.22	5.31	5.23	5.27
(11)	1	2264	1995	2.09	2.05	2.06	2.06

Table 6 presents the results of the calculations with Timoshenko effect and Bernstein double estimators by formulas (22) for the boundary conditions (8), then by (26) for the conditions (11), and the results on the base of biquadratic equation (32) for the case (10). The results of calculations for a rough Dunkerley estimator for cases (8), (10) and (11) were based on the formula (JAROSZEWICZ, ZORYJ 2000):

$$\tilde{\omega}_1 \approx \frac{1}{\sqrt{a_1}} \tag{38}$$

where $\tilde{\omega}_1$ is the course value of the basic frequency estimate, calculated on the basis of the first part of the series a_1.

For the case of a free-ends beam without Timoshenko effect, on the base of (32) and Table 3 formulas for the cases (8)÷(11) can be formulated:

$$\tilde{\omega}_1 \approx \frac{1}{\sqrt{B/420}},\ \tilde{\omega}_1 \approx \frac{1}{\sqrt{B/720}},\ \tilde{\omega}_1 \approx \frac{1}{\sqrt{B/90}},\ \tilde{\omega}_1 \approx \frac{1}{\sqrt{B/12}} \tag{39}$$

Numerical analysis

For comparison purposes, the calculations of the distribution and values of concentrated strain have been calculated with the use of FEM. Autodesk Inventor 2015 software equipped with NASTRAN module has been applied. The comparison has only related to natural frequency distribution determined analytically using the relationship (31) and (38), so the focus was on the calculation with linear behavior of the material. Calculations covered four cases of beam support shown in Table 2. The dimensions of the beam, material constants and boundary conditions were assumed in the same way as in the analytical studies. Due to the simple shape of the beam, the problem was modeled as a spatial one, adopting a standard mesh that uses 10-node elements. Table 7 shows information on the developed mesh for the studied beam.

Calculations of the percentage difference in the results for different types of beam supports and calculation methods have been conducted according to the formula (40), and these values are given in Table 8.

$$\Delta s = \left| \frac{{\omega_{sr}}^{ANL} - \omega^{MES}}{{\omega_{sr}}^{ANL}} \right| \cdot 100\% \tag{40}$$

where:

${\omega_{sr}}^{ANL}$ – analytically determined natural frequency,

ω^{MES} – FEM-determined natural frequency.

Table 7

Mesh parameters

Type of mesh	Parabolic tetrahedral
Max. element Growth Rate	1.5
Refinement Ratio	0.2
Min./Max. Trangle Angle	20/30 deg
Upper Jacobian Ratio Bound	16 points
Size	200 mm
Tolerance	0.0356 mm
Quality	high
Elements	2,600
Nodes	3,851

Table 8

Percentage values of difference in results for FEM and analytical method

Boundary conditions	Values ω^{MES} [rad/s]		Difference in results without Timoshenko effect [%]		Difference in results with Timoshenko effect [%]	
	1st freq.	2nd freq.	1st freq.	2nd freq.	1st freq.	2nd freq.
(8)	12.87	29.96	5.28	8.71	2.04	4.23
(9)	8.53	23.41	4.31	7.76	3.47	3.12
(10)	5.41	21.541	2.88	6.48	2.65	5.08
(11)	2.08	11.78	2.15	5.37	1.17	2.81

Conclusion

The characteristic equations which have been obtained in this paper and final formulas resulting from them, provide the mathematical safeguard for calculation of the lower and upper estimators of the base frequencies of short beams with the rotational inertia and non-dilatational strain taken into account, and with the most common boundary conditions found in engineering practice.

Application of the Cauchy function and the corresponding characteristic series for analysis of transverse vibrations of short beams has demonstrated efficiency of these methods in terms of accuracy and in the ability of formulating functional relations between the mass and elasticity parameters of beams for an arbitrary section, arbitrary relationship of longitudinal and non-dilatational elasticity module, and arbitrary natural frequencies.

Consideration of first three terms of series allows for receiving convergence to the analytical solution results of the fundamental frequency coefficients, with accuracy to three decimal digits.

Formulas, proposed in this paper can be particularly useful for preliminary engineering calculations as well as for didactic objectives in the dynamics of short structural elements, in which application of simplified technical bending theory leads to unacceptable errors in natural frequency calculations, especially for higher frequencies, starting from the second one.

FEM – Autodesk NASTRAN environment allows for precise forecasting of a beam's natural frequencies. Frequencies whose values are, for the most part, within the specified analytical considerations range. It can be observed that the results obtained in the FEM environment show lower percentage differences with respect to the analytical considerations when the Timoshenko effect is taken into account.

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