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IRWIN CRITERION FOR A CRACK OF FINITE CONDUCTIVITY IN AN ELASTIC DIELECTRIC

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Abstract

The paper is a continuation of the investigations presented shortly in a previous article (KURLANDZKA 2005), which focused on solving of a boundary-value problem for an electro-elastic field in the vicinity of a tip of the crack of finite conductivity in an elastic, isotropic and homogeneous dielectric. Stress distribution in the vicinity of the crack tip, mechanical stress intensity coefficients, electric stress intensity coefficients and the intensity coefficient of an electric field were defined in the present study. Based on the generalized Griffith criterion, the Irwin generalized criterion for the case considered was derived.

KRYTERIUM IRWINA DLA SZCZELINY O SKOŃCZONEJ PRZEWODNOŚCI W DIELEKTRYKU SPRĘŻYSTYM

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Słowa kluczowe: pękanie, pole elektryczne, dielektryk sprężysty, współczynniki intensywności.

Streszczenie

Praca jest kontynuacją badań przedstawionych krótko we wcześniejszej publikacji (KURLANDZKA 2005). Podano tam rozwiązanie problemu brzegowego dla pól elektro-sprężystych w otoczeniu wierzchołka szczeliny o skończonej przewodności w jednorodnym i izotropowym dielektryku sprężystym. W prezentowanej pracy przedstawiono rozkład naprężeń w otoczeniu wierzchołka szczeliny, zdefiniowano współczynniki naprężeń mechanicznych i elektrycznych oraz współczynnik intensywności pola elektrycznego. Opierając się na uogólnionym kryterium Griffitha oraz uzyskanych dla rozpatrywanego przypadku wynikach, otrzymano uogólnione kryterium Irwina.

Introduction

The aim of the paper was to extend the Irwin criterion to the case of a crack of finite conductivity in an elastic, isotropic and homogeneous dielectric.

If a crack in an dielectric influenced by an external electric field is considered, the electric properties of the media inside the crack should be determined. It seems natural if we assume that the crack is filled with air which, if the intensity of the electric field is not high, behaves like vacuum. However, if the intensity of the electric field attains ionization intensity, which for air is about $2 \div 3 \cdot 10^4$ *V*/cm, the gas inside the crack behaves like a perfect conductor. This changes considerably the behavior of the electro-elastic fields in the vicinity of the crack tip. The electric field and electric stresses, which in the case of a vacuum crack have no singularity in the crack tip, become singular and contribute directly to energy, which can be used for the creation of a new crack surface and, in consequence, a fracture of the material.

However, between these two extreme cases there must be a state in which the gas inside the crack behaves like a conductor of finite conductivity. This case is the subject of the present investigations. Both electric resistance and heat effects are related to finite conductivity. Thus, it must be analyzed whether the generalized Griffith criterion, providing the basis for investigations of the influence the electric field on crack propagation, should be modified.

The Irwin criterion for the case of a perfectly conducting crack in an elastic dielectric was presented in a previous paper (KURLANDZKA 2005). This criterion, derived from the Griffith criterion generalized to the case of electro-elastic interactions (KURLANDZKA 2005, 1998), regarded the case of vacuum and a perfectly conducting crack in an elastic dielectric with no conductivity. Thus, there were no reasons to take Joule-Lenz heating into account. In this study the case of a crack of finite conductivity is considered and therefore it must be decided whether the criterion needs modification.

Let us remind the generalized Griffith criterion for the case of a dielectric containing a vacuum crack or a perfectly conducting crack, assuming that crack increment takes place along the x^1 axis of the Cartesian coordinate system:

$$-\frac{1}{2}\lim_{\xi\to 0}\int_{S_\xi}\Big\{\big[(\sigma_{kl}+t_{kl}+\tau_{kl})u_{k;1}+d_l\phi_{;1}\big]n_l + -\left[\rho(\Sigma+\eta)-\frac{1}{2}\varepsilon\phi_{;k}\phi_{;k}\right]n_1\Big\}ds=\gamma \tag{1}$$

where:

S_ξ – cylindrical surface of radius ξ surrounding the crack edge,
γ – material parameter called surface energy,
$\boldsymbol{\sigma}$, $\boldsymbol{t}$, $\boldsymbol{\tau}$ – mechanical, electro-elastic and electric parts of the stress tensor respectively,
u – displacement vector,
d – electric induction,
ϕ – electric potential,
ε – dielectric permittivity,
Σ – elastic energy,
η – energy of coupling of electric and elastic interactions,
n – unit normal vector external to S_ξ.

The dielectric is non-conductive, only the gas inside the crack behaves like a conductor of finite conductivity. The surface S_ξ surrounding the crack tip is closed and as its diameter tends to zero, a contribution to the expression on the left side is only due to the singular parts of the functions appearing in the integrand. Let us remind the general solution for the electric field and potential (KURLANDZKA 2005).

The electric potential and electric field in the dielectric:

$$\phi_d=-\sum_{n=0}r^n\frac{c_n}{n}\cos n\alpha-2\sum_{n=1}r^{\frac{n}{2}}\frac{d_n}{n}\sin\frac{n}{2}\alpha \tag{2}$$

$$\begin{aligned}E_r&=\sum_{n=1}^{\infty}\left(r^{n-1}c_n\cos n\alpha+r^{\frac{n}{2}-1}d_n\sin\frac{n}{2}\alpha\right)\\ E_\alpha&=\sum_{n=1}^{\infty}\left(-r^{n-1}c_n\sin n\alpha+r^{\frac{n}{2}-1}d_n\cos\frac{n}{2}\alpha\right)\end{aligned} \tag{3}$$

The electric potential and electric field inside the crack:

$$\phi_c=-\sum_{n=0}r^n\frac{c_n}{n} \tag{4}$$

$$\breve{E}_r = \sum_{n=1}^{\infty} r^{n-1} c_n, \quad \breve{E}_\alpha = 0 \tag{5}$$

where:

r, α – polar coordinates connected with the Cartesian coordinates x^1, x^2 by means of the relations:

$$x^1 = r\cos\alpha, \quad x^2 = r\sin\alpha$$

The solution satisfies the equations in the vicinity of the crack tip Ω: $r < r_0$, $0 < \alpha < 2\pi$ and the corresponding boundary conditions on the crack surface

$$S_c: \; r < r_0, \quad \alpha = \begin{cases} 0 \\ 2\pi \end{cases}$$

c_n, d_n are arbitrary constants that can be determined from the conditions on the boundary of the domain $r = r_0$, $0 < \alpha < 2\pi$. They depend on the external electric field in the outside domain $r > r_0$.

There is no singularity in the potential and electric field inside the crack. Thus, there is no contribution to energy which could be used for crack length increment, and so the Griffith criterion needs no modification in the case considered.

The distribution of the fields present in the integrand, with accuracy to the terms contributing to fracture energy, will be given in the next section. The functions will be used further in the procedure of deriving of the generalized Irwin criterion.

Let us keep in mind that the solution of the boundary-value problem for a crack of finite conductivity in an elastic isotropic and homogeneous dielectric is obtained for a quasi-linear approximation of the Toupin-Eringen (TOUPIN 1956, ERINGEN 1962) nonlinear model of the dielectric, with some modifications by the author (KURLANDZKA 1998). The approximation is obtained assuming small strains and strong electric intensity. These assumptions lead to the equations and the boundary conditions for an electric field in a rigid body and the Lamé equations and the corresponding boundary conditions in which electric stresses, nonlinear with respect to electric intensity, appear. The solution is general, satisfies equations in the neighborhood of the crack tip, the corresponding boundary conditions on the crack surface and the condition of finite energy at the crack tip and includes a number of arbitrary constants which can be determined from the corresponding conditions on the boundary of the considered domain.

Electro-elastic fields in the vicinity of the crack tip

The functions describing the electric potential and intensity are given in the previous section. Assuming that the series present in the solution are uniformly convergent, the electric stresses take the form:

$$t_{rr}+\tau_{rr}=r^{-\frac{1}{2}}d_1\left\{c_1\left[-\left(a_5+2a_2\right)\sin\frac{\alpha}{2}+\left(a_5+\varepsilon\right)\sin\frac{3}{2}\alpha\right]+\right.$$
$$\left.+d_2\left[\left(a_5+2a_2\right)\cos\frac{\alpha}{2}-\left(a_5+\varepsilon\right)\cos\frac{3}{2}\alpha\right]\right\}+O\left(r^0\right)$$

$$t_{r\alpha}+\tau_{r\alpha}=r^{-\frac{1}{2}}d_1\left(a_5+\varepsilon\right)\left[c_1\cos\frac{3}{2}\alpha+d_2\sin\frac{3}{2}\alpha\right]+O\left(r^0\right) \tag{6}$$

$$t_{\alpha\alpha}+\tau_{\alpha\alpha}=r^{-\frac{1}{2}}d_1\left\{-c_1\left[\left(2a_2+a_5\right)\sin\frac{\alpha}{2}+\left(a_5+\varepsilon\right)\sin\frac{3}{2}\alpha\right]+\right.$$
$$\left.+d_2\left[\left(2a_2+a_5\right)\cos\frac{\alpha}{2}+\left(a_5+\varepsilon\right)\cos\frac{3}{2}\alpha\right]\right\}+O\left(r^0\right)$$

where

a_2, a_5 – material constants,

c_1, d_1, d_2– arbitrary constants present in the solution for an electric field.

If the arbitrary constants they are determined from the conditions on the common boundary of the neighborhood of the crack tip and the outside domain, the electric field and electric stresses are uniquely determined.

Now let us remind the general solution for the displacements generated by the electric field (KURLANDZKA 2005):

$$u_r=r^{\frac{1}{2}}\left[C_1\left(\frac{\lambda+5\mu}{\lambda+\mu}\sin\frac{\alpha}{2}+\sin\frac{3}{2}\alpha\right)+D_1\left(\frac{\lambda+5\mu}{\lambda+\mu}\cos\frac{\alpha}{2}+3\cos\frac{3}{2}\alpha\right)+\right.$$
$$+c_1d_1\left[\left(-\frac{a_5+2a_2}{3\lambda+7\mu}\right)\sin\frac{\alpha}{2}+\frac{1}{\mu}\left(2\left(a_5+2a_2\right)\frac{\mu\left(\lambda+3\mu\right)}{\left(3\lambda+7\mu\right)\left(\lambda+5\mu\right)}-a_5-\varepsilon\right)\sin\frac{3}{2}\alpha\right]+$$
$$\left.+d_1d_2\left[\frac{a_5+2a_2}{3\lambda+7\mu}\cos\frac{\alpha}{2}+\frac{1}{\mu}\left(2\left(a_5+2a_2\right)\frac{\left(\lambda+2\mu\right)\left(3\lambda+13\mu\right)}{\left(3\lambda+7\mu\right)\left(\lambda+5\mu\right)}+a_5+\varepsilon\right)\right]\cos\frac{3}{2}\alpha\right]+$$
$$+O\left(r^1\right) \tag{7}$$

$$u_{\alpha} = r^{\frac{1}{2}}\Big[C_1\left(\frac{3\lambda+7\mu}{\lambda+\mu}\cos\frac{\alpha}{2}+\cos\frac{3}{2}\alpha\right)-D_1\left(\frac{3\lambda+7\mu}{\lambda+\mu}\sin\frac{\alpha}{2}+3\sin\frac{3}{2}\alpha\right)+$$
$$+c_1d_1\left[\frac{a_5+2a_2}{\lambda+5\mu}\cos\frac{\alpha}{2}+\frac{1}{\mu}\left(2\left(a_5+2a_2\right)\frac{\mu(\lambda+3\mu)}{(3\lambda+7\mu)(\lambda+5\mu)}-a_5-\varepsilon\right)\cos\frac{3}{2}\alpha\right]+$$
$$+d_1d_2\left[\frac{a_5+2a_2}{\lambda+5\mu}\sin\frac{\alpha}{2}-\frac{1}{\mu}\left(2\left(a_5+2a_2\right)\frac{(\lambda+2\mu)(3\lambda+13\mu)}{(3\lambda+7\mu)(\lambda+5\mu)}+a_5+\varepsilon\right)\sin\frac{3}{2}\alpha\right]\Big]+$$
$$+O\left(r^1\right) \tag{8}$$

where:

λ, μ – Lamé constants. The functions include the arbitrary constants C_1, D_1 which can be determined from the corresponding conditions on the common boundary of the neighborhood of the crack tip and the outside domain, as well as the arbitrary constants connected directly with the electric field c_1, d_1, d_2.

Inserting the above functions into the constitutive relations for mechanical stresses, we get functions describing mechanical stresses in the vicinity of the crack tip in the following form:

$$\sigma_{rr} = r^{-\frac{1}{2}}\Big\{\mu\left[C_1\left(5\sin\frac{\alpha}{2}+\sin\frac{3}{2}\alpha\right)+D_1\left(5\cos\frac{\alpha}{2}+\cos\frac{3}{2}\alpha\right)\right]+$$
$$+c_1d_1\Big[-\left(a_5+2a_2\right)\frac{3\lambda^2+12\lambda\mu+5\mu^2}{(\lambda+5\mu)(3\lambda+7\mu)}\sin\frac{\alpha}{2}+$$
$$+\left(2\left(a_5+2a_2\right)\frac{\mu(\lambda+3\mu)}{(\lambda+5\mu)(3\lambda+7\mu)}-a_5-\varepsilon\right)\sin\frac{3}{2}\alpha\Big]+$$
$$+d_1d_2\Big[\left(a_5+2a_2\right)\frac{3\lambda^2+12\lambda\mu+5\mu^2}{(\lambda+5\mu)(3\lambda+7\mu)}\cos\frac{\alpha}{2}+$$
$$+\left[2\left(a_5+2a_2\right)\frac{(\lambda+2\mu)(3\lambda+13\mu)}{(3\lambda+7\mu)(\lambda+5\mu)}+a_5+\varepsilon\right]\cos\frac{3}{2}\alpha\Big]\Big\}+$$
$$+O\left(r^0\right)$$

$$\begin{aligned}\sigma_{r\alpha} = r^{-\frac{1}{2}}\Big\{&\mu\left[C_1\left(-\cos\frac{\alpha}{2}+\cos\frac{3}{2}\alpha\right)+D_1\left(\sin\frac{\alpha}{2}-3\sin\frac{3}{2}\alpha\right)\right]+\\
&+c_1d_1\Big[-2(a_5+2a_2)\frac{\mu(\lambda+3\mu)}{(\lambda+5\mu)(3\lambda+7\mu)}\cos\frac{\alpha}{2}+\\
&+\left(2(a_5+2a_2)\frac{\mu(\lambda+3\mu)}{(\lambda+5\mu)(\lambda+5\mu)}-a_5-\varepsilon\right)\cos\frac{3}{2}\alpha\Big]+\\
&-d_1d_2\Big[2(a_5+2a_2)\frac{\mu(\lambda+3\mu)}{(\lambda+5\mu)(3\lambda+7\mu)}\sin\frac{\alpha}{2}+\\
&+\left(2(a_5+2a_2)\frac{(\lambda+2\mu)(3\lambda+13\mu)}{(3\lambda+7\mu)(\lambda+5\mu)}+a_5+\varepsilon\right)\sin\frac{3}{2}\alpha\Big]\Big\}+\\
&+O\left(r^0\right)\end{aligned} \tag{9}$$

$$\begin{aligned}\sigma_{\alpha\alpha} = r^{-\frac{1}{2}}\Big\{&\mu\left[C_1\left(3\sin\frac{\alpha}{2}-\sin\frac{3}{2}\alpha\right)+3D_1\left(\cos\frac{\alpha}{2}-\cos\frac{3}{2}\alpha\right)\right]+\\
&-c_1d_1\Big[(a_5+2a_2)\frac{3\lambda^2+16\lambda\mu+17\mu^2}{(\lambda+5\mu)(3\lambda+7\mu)}\sin\frac{\alpha}{2}+\\
&+\left(2(a_5+2a_2)\frac{\mu(\lambda+3\mu)}{(\lambda+5\mu)(3\lambda+7\mu)}-a_5-\varepsilon\right)\sin\frac{3}{2}\alpha\Big]+\\
&+d_1d_2\Big[(a_5+2a_2)\frac{3\lambda^2+16\lambda\mu+17\mu^2}{(\lambda+5\mu)(3\lambda+7\mu)}\cos\frac{\alpha}{2}+\\
&-\left(2(a_5+2a_2)\frac{(\lambda+2\mu)(3\lambda+13\mu)}{(3\lambda+7\mu)(\lambda+5\mu)}+a_5+\varepsilon\right)\cos\frac{3}{2}\alpha\Big]\Big\}+\\
&+O\left(r^0\right)\end{aligned} \tag{10}$$

The mechanical stresses include parts dependent directly on the electric field in the neighborhood of the crack tip, and parts known from fracture mechanics, dependent on the arbitrary constants C_1, D_1.

Intensity coefficients

The definitions of the intensity coefficients generalized to the case of electromagnetic fracture are given in (KURLANDZKA 2005) for the case of the vacuum and perfectly conducting crack. Now they will be given for the considered case of the conducting crack of finite conductivity.

Taking into account formulae (9), (10), mechanical stress intensity coefficients are obtained in the form:

$$
\begin{aligned}
{K_I}^M &= \lim_{r\to 0}\sqrt{2\pi r}\,\sigma_{\alpha\alpha}(r,\pi) = {K_I}^{MC} + {K_I}^{ME} \\
{K_{II}}^M &= \lim_{r\to 0}\sqrt{2\pi r}\,\sigma_{r\alpha}(r,\pi) = {K_{II}}^{MC} + {K_{II}}^{ME}
\end{aligned}
\tag{11}
$$

where

$$
{K_I}^{MC} = 4\mu C_1, \quad {K_{II}}^{MC} = 4\mu D_1
$$

are the same as the stress intensity coefficients in fracture mechanics. The electric parts of the mechanical stress intensity coefficients are

$$
\begin{aligned}
{K_I}^{ME} &= \frac{-\sqrt{2\pi}c_1 d}{(\lambda+5\mu)(3\lambda+7\mu)} \\
&\quad \left[(a_5+2a_2)(\lambda+\mu)(3\lambda+11\mu)+(a_5+\varepsilon)(\lambda+5\mu)(3\lambda+7\mu)\right] \\
{K_{II}}^{ME} &= \frac{\sqrt{2\pi}d_1 d_2}{(\lambda+5\mu)(3\lambda+7\mu)} \\
&\quad \left[2(a_5+2a_2)\left(3\lambda^2+18\lambda\mu+23\mu^2\right)+(a_5+\varepsilon)(\lambda+5\mu)(3\lambda+7\mu)\right]
\end{aligned}
\tag{12}
$$

They depend directly on the arbitrary constants present in the solution for the electric field.

The electric stress intensity coefficients obtained on the basis of the definitions and formulae (6) are:

$$
\begin{aligned}
{K^E}_I &= \lim_{r\to 0}\sqrt{2\pi r}\left[t_{\alpha\alpha}(r,\pi)+\tau_{\alpha\alpha}(r,\pi)\right] = \sqrt{2\pi}c_1 d_1(\varepsilon-2a_2) \\
{K^E}_{II} &= \lim_{r\to 0}\sqrt{2\pi r}\left[t_{r\alpha}(r,\pi)+\tau_{r\alpha}(r,\pi)\right] = -\sqrt{2\pi}d_1 d_2(a_5+\varepsilon)
\end{aligned}
\tag{13}
$$

Let us notice that the electric parts of the mechanical stress intensity coefficients and the electric stress intensity coefficients depend on the arbitrary constants present in the solution for the electric field c_1, d_1, d_2. Hence, the electric parts of the mechanical stress intensity coefficients can be expressed by means of the electric stress intensity coefficients:

$$
\begin{aligned}
K_I^{ME} &= -\frac{(a_5+2a_2)(\lambda+\mu)(3\lambda+11\mu)+(a_5+\varepsilon)(\lambda+5\mu)(3\lambda+7\mu)}{(\varepsilon-2a_2)(\lambda+5\mu)(3\lambda+7\mu)} K_I^E \\
K_{II}^{ME} &= -\frac{2(a_5+2a_2)\left(3\lambda^2+18\lambda\mu+23\mu^2\right)+(a_5+\varepsilon)(\lambda+5\mu)(3\lambda+7\mu)}{(a_5+\varepsilon)(\lambda+5\mu)(3\lambda+7\mu)} K_{II}^E
\end{aligned}
\tag{14}
$$

The displacements on the crack surface and stresses in the material on prolongation of the crack can be expressed by means of the generalized stress intensity coefficients:

$$
\begin{aligned}
u_r(r,0) = -u_r(r,2\pi) = \frac{1}{\mu}\sqrt{\frac{r}{2\pi}}\Bigg\{ & K_{II}^{MC}\frac{\lambda+2\mu}{\lambda+\mu} + \\
& +K_{II}^{ME}\frac{3(a_5+2a_2)\left(2\lambda^2+13\lambda\mu+19\mu^2\right)+(a_5+\varepsilon)(\lambda+5\mu)(3\lambda+7\mu)}{2(a_5+2a_2)\left(3\lambda^2+18\lambda\mu+23\mu^2\right)+(a_5+\varepsilon)(\lambda+5\mu)(3\lambda+7\mu)}\Bigg\} + \\
& +O\left(r^1\right)
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
u_\alpha(r,0) = -u_\alpha(r,2\pi) = \frac{1}{\mu}\sqrt{\frac{r}{2\pi}}\Bigg\{ & K_I^{MC}\frac{\lambda+2\mu}{\lambda+\mu} + \\
& -K_{II}^{ME}\frac{(a_5+2a_2)\mu(5\lambda+13\mu)-(a_5+\varepsilon)(\lambda+5\mu)(3\lambda+7\mu)}{(a_5+2a_2)(\lambda+\mu)(3\lambda+11\mu)+(a_5+\varepsilon)(\lambda+5\mu)(3\lambda+7\mu)}\Bigg\} + \\
& +O\left(r^1\right)
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
\left(\sigma_{r\alpha}+t_{r\alpha}+\tau_{r\alpha}\right)_{|\alpha=\pi} &= \frac{1}{\sqrt{2\pi r}}\left(K_{II}^{MC}+K_{II}^{ME}+K_{II}^{E}\right)+O\left(r^0\right) \\
\left(\sigma_{\alpha\alpha}+t_{\alpha\alpha}+\tau_{\alpha\alpha}\right)_{|\alpha=\pi} &= \frac{1}{\sqrt{2\pi r}}\left(K_{I}^{MC}+K_{I}^{ME}+K_{I}^{E}\right)+O\left(r^0\right)
\end{aligned}
\tag{17}
$$

The electric field intensity coefficients according to the definition (KURLANDZKA 1998, 2005) are:

$$I = -\frac{\sqrt{\pi}}{2} \lim_{r\to 0} r^{-\frac{1}{2}} \phi_d \left(r,\pi\right) = \sqrt{\pi} d_1.$$

$$I_c = -\frac{\sqrt{\pi}}{2} \lim_{r\to 0} r^{-\frac{1}{2}} \phi_c \left(r,\pi\right) = 0.$$

The electric potential in a dielectric on prolongation of the crack and the normal component of electric induction on the crack surface can be expressed by means of the electric field intensity coefficient in the following form:

$$\begin{aligned} \phi_d\left(r,\pi\right) &= -2I\sqrt{\frac{r}{\pi}} + O\left(r^1\right) \\ D_\alpha\left(r,0\right) &= -D_\alpha\left(r,2\pi\right) = \frac{\varepsilon I}{\sqrt{\pi r}} + O\left(r^0\right) \end{aligned} \tag{18}$$

The above formulae will be used in the procedure of deriving the generalized Irwin criterion for the case considered.

Irwin criterion

The starting point for the derivation of the electric fracture criterion in the form of the generalized Irwin criterion is the Griffith criterion (1). Let us introduce two Cartesian coordinates systems: stationary frame (x_1, x_2) with the origin located at the crack tip before elongation, and moving frame ($x_1{}'$, $x_2{}'$) whose origin is also located at the crack tip, but moves together with the crack tip.

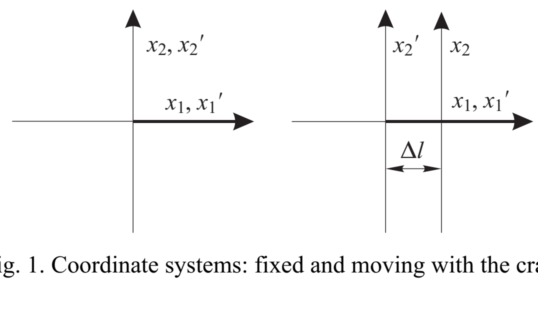

Fig. 1. Coordinate systems: fixed and moving with the crack

The coordinates of the points in the stationary and moving frame are interrelated by means of the formula:

$$x_1' = x_1 + \Delta l,\ x_2' = x_2.$$

The integral in (1) is taken over the surface of the tube surrounding the crack edge and moving with the crack. The energy used for the creation of a new surface of the crack is equal to the limit value of the integral, when the diameter of the tube tends to zero. The limit value of the integral is finite. It results from the energy existence condition imposed on the solution of the problem in the vicinity of the crack tip (KURLANDZKA 2005, 1998). Then the contour of integration is arbitrary and the iterative limits are equal to the limit in the common sense. Let us take the contour as S_ξ:

$$S_\xi : -a \le x_1' \le a,\ -\xi \le x_2' \le \xi.$$

If $\xi \to 0$ surface S_ξ tends to a two-sided segment

$$-a \le x_1' \le a, \quad x_2' = \pm 0.$$

Let us notice that a contribution to the limit is only due to the singular parts of the integrand. The functions in the integrand have singularity at the crack tip and there are no singularity on the vertical lines $x_1' = -a,\ x_1' = a$. Thus, the limit integration over the vertical lines gives no contribution to the energy used for creating the new crack surface.

Moreover, before the crack increment for $x_1 > 0$

$$\sigma_{k2}(x_1, \pm 0) + t_{k2}(x_1, \pm 0) + \tau_{k2}(x_1, \pm 0) = 0,$$
$$\phi_d(x_1, \pm 0) = \phi_d(r, \alpha)\Big|_{r = x_1, \alpha = \left\{\begin{smallmatrix} 0 \\ 2\pi \end{smallmatrix}\right.} = 0 + O\left(r^1\right).$$

Taking into account the definition of the derivative with respect to x_1 and the fact that the increment of the respective functions is generated by crack elongation, the criteria can be written in the stationary frame in the form:

$$\gamma = -\frac{1}{2}\lim_{a\to 0}\lim_{\Delta l \to 0}\frac{1}{\Delta l}\int_{-a-\Delta l}^{0}\{[\sigma_{k2}(x_1 + \Delta l, +0) + t_{k2}(x_1 + \Delta l, +0) +$$
$$+\tau_{k2}(x_1 + \Delta l, +0)]\Delta u_k(x_1 + \Delta l, +0) + D_2(x_1 + \Delta l, +0)\Delta\phi_d(x_1 + \Delta l, +0) +$$
$$+\left[\sigma_{k2}(x_1 + \Delta l, -0) + t_{k2}(x_1 + \Delta l, -0) + \tau_{k2}(x_1 + \Delta l, -0)\right]\Delta u_k(x_1 + \Delta l, -0) +$$
$$+D_2(x_1 + \Delta l, -0)\Delta\phi_d(x_1 + \Delta l, -0)\}dx_1'$$

where $\Delta\mathbf{u}$ and $\Delta\phi_d$ are the increments of respective functions due to crack elongation by Δl.

Let us introduce a denotation:

$$\mathbf{s} = \boldsymbol{\sigma} + \mathbf{t} + \boldsymbol{\tau},\ K_I = K_I{}^{MC} + K_I{}^{ME} + K_I{}^{E},\ K_{II} = K_{II}{}^{MC} + K_{II}{}^{ME} + K_{II}{}^{E}.$$

All the functions appearing in the integrand can be determined from formulae (2), (3), (6)–(10), taking into account the formulae of transformation from polar to Cartesian coordinates, with the use of the following relations:

– Before crack elongation: $x_1' = x_1 + \text{Ä}l = x_1$

$-\alpha - \Delta l \le x_1 \le 0$	$x_1 > 0$
$s_{12}(x_1,\pm 0) = \left[s_{r\alpha}(r,\alpha)\right]_{r=\lvert x_1\rvert,\alpha=\pm\pi}$ $= \dfrac{\pm 1}{\sqrt{2\pi\lvert x_1\rvert}}(K_{II}) + O\left(\lvert x_1\rvert^0\right)$	$s_{12}(x_1,\pm 0) = \left[s_{r\alpha}(r,\alpha)\right]_{r=x_1,\alpha=\begin{cases}0\\2\pi\end{cases}} = 0$
$s_{22}(x_1,\pm 0) = \left[s_{\alpha\alpha}(r,\alpha)\right]_{r=\lvert x_1\rvert,\alpha=\pm\pi}$ $= \dfrac{\pm 1}{\sqrt{2\pi\lvert x_1\rvert}}(K_I) + O\left(\lvert x_1\rvert^0\right)$	$s_{22}(x_1,\pm 0) = \left[s_{\alpha\alpha}(r,\alpha)\right]_{r=x_1,\alpha=\begin{cases}0\\2\pi\end{cases}} = 0$
$D_2(x_1,\pm 0) = -D_\alpha(r,\alpha)\big\rvert_{r=\lvert x_1\rvert,\alpha=\pm\pi} = 0$	$D_2(x_1,\pm 0) = D_\alpha(r,\alpha)\big\rvert_{r=x_1,\alpha=\begin{cases}0\\2\pi\end{cases}}$ $= \pm\dfrac{\varepsilon I}{\sqrt{\pi x_1}} + O\left((x_1)^0\right)$
$\Delta u = 0,\ \Delta\phi_d = 0$	$\Delta u = 0,\ \Delta\phi_d = 0$
$\phi_d(x_1,\pm 0) = \phi_d(r,\alpha)\big\rvert_{r=\lvert x_1\rvert,\alpha=\pm\pi}$ $= \mp 2I\sqrt{\dfrac{\lvert x_1\rvert}{\pi}} + O\left(\lvert x_1\rvert^1\right)$	$\phi_d(x_1,\pm 0) = \phi_d(r,\alpha)\big\rvert_{r=x_1,\alpha=\begin{cases}0\\2\pi\end{cases}} = 0$

– After crack increment by Δl: $x_1 = x_1' - \Delta l$

$-\alpha - \Delta l \le x_1 < -\Delta l$	$-\Delta l \le x_1 \le 0$
$s_{12}(x_1 + \Delta l,\pm 0) = \left[s_{r\alpha}(r,\alpha)\right]_{r=\lvert x_1\rvert-\Delta l,\alpha=\pm\pi}$ $= \dfrac{\pm K_{II}}{\sqrt{2\pi(\lvert x_1\rvert - \Delta l)}} + O\left[(\lvert x_1\rvert - \Delta l)^0\right]$	$s_{12}(x_1,\pm 0) = \left[s_{r\alpha}(r,\alpha)\right]_{r=\Delta l-\lvert x_1\rvert,\alpha=\begin{cases}0\\2\pi\end{cases}} = 0$

$s_{22}(x_1+\Delta l,\pm 0)=[s_{\alpha\alpha}(r,\alpha)]_{r=\lvert x_1\rvert-\Delta l,\alpha=\pm\pi}$ $=\frac{\pm 1}{\sqrt{2\pi(\lvert x_1\rvert-\Delta l)}}(K_I)+O\left[(\lvert x_1\rvert-\Delta l)^0\right]$	$s_{22}(x_1,\pm 0)=[s_{\alpha\alpha}(r,\alpha)]_{r=\Delta l-\lvert x_1\rvert,\alpha=\begin{cases}0\\2\pi\end{cases}}=0$
$u_1(x_1+\Delta l,\pm 0)=-u_r(r,\alpha)\rvert_{r=\lvert x_1\rvert-\Delta l,\alpha=\pm\pi}$	$u_1(x_1+\Delta l,\pm 0)=u_r(r,\alpha)\rvert_{r=\Delta l-\lvert x_1\rvert,\alpha=\begin{cases}0\\2\pi\end{cases}}$
$u_2(x_1+\Delta l,\pm 0)=-u_\alpha(r,\alpha)\rvert_{r=\lvert x_1\rvert-\Delta l,\alpha=\pm\pi}$	$u_2(x_1+\Delta l,\pm 0)=u_\alpha(r,\alpha)\rvert_{r=\Delta l-\lvert x_1\rvert,\alpha=\begin{cases}0\\2\pi\end{cases}}$
$D_2(x_1+\Delta l,\pm 0)=-D_\alpha(r,\alpha)\rvert_{r=\lvert x_1\rvert-\Delta l,\alpha=\pm\pi}=0$	$D_2(x_1,\pm 0)=D_\alpha(r,\alpha)\rvert_{r=\Delta l-\lvert x_1\rvert,\alpha=\begin{cases}0\\2\pi\end{cases}}$ $=\pm\frac{\varepsilon I}{\sqrt{\pi(\Delta l-\lvert x_1\rvert)}}+O\left[(\Delta l-\lvert x_1\rvert)^0\right]$
$\phi_d(x_1+\Delta l,\pm 0)=\phi_d(r,\alpha)\rvert_{r=\lvert x_1\rvert-\Delta l,\alpha=\pm\pi}$ $=\mp 2I\sqrt{\frac{\lvert x_1\rvert-\Delta l}{\pi}}+O\left[(\lvert x_1\rvert-\Delta l)^1\right]$	$\phi_d(x_1+\Delta l,\pm 0)=\phi_d(r,\alpha)\rvert_{r=\delta l-\lvert x_1\rvert,\alpha=\begin{cases}0\\2\pi\end{cases}}=0$

The functions $u_r(r,\alpha)\Big|_{r=\Delta l-|x_1|,\alpha=\begin{cases}0\\2\pi\end{cases}}$ and $u_\alpha(r,\alpha)\Big|_{r=\Delta l-|x_1|,\alpha=\begin{cases}0\\2\pi\end{cases}}$ are given by means of (15)–(16).

Taking the above into account, the criterion is reduced to the following relation:

$$\gamma=-\lim_{\Delta l\to 0}\frac{1}{\Delta l}\int_{-\Delta l}^{0}\Big\{\big[\sigma_{k2}(x_1+\Delta l,+0)+t_{k2}(x_1+\Delta l,+0)+\tau_{k2}(x_1+\Delta l,+0)\big]$$
$$\cdot\big[u_k(x_1+\Delta l,+0)-u_k(x_1,+0)\big]+D_2(x_1+\Delta l,+0)\big[\phi_d(x_1+\Delta l,+0)-\phi_d(x_1,+0)\big]\Big\}dx_1$$

Following the Irwin procedure, which is equivalent to the assumption $\Delta u_k\approx u_k(x_1+\Delta l,+0)$, after inserting the values of respective functions into the integrand, integration and the limit procedure, the generalized Irwin criterion is obtained in the following form:

$$\gamma = \frac{1}{4\mu}\left\{\frac{\lambda+2\mu}{\lambda+\mu}\left[K_I{}^{MC}\left(K_I{}^{MC}+K_I{}^{ME}+K_I{}^{E}\right)+K_{II}{}^{MC}\left(K_{II}{}^{MC}+K_{II}{}^{ME}+K_{II}{}^{E}\right)\right]+\right.$$
$$-\frac{(a_5+2a_2)\mu(5\lambda+13\mu)-(a_5+\varepsilon)(\lambda+5\mu)(3\lambda+7\mu)}{(a_5+2a_2)(\lambda+\mu)(3\lambda+11\mu)+(a_5+\varepsilon)(\lambda+5\mu)(3\lambda+7\mu)}\left(K_I{}^{MC}+K_I{}^{ME}+K_I{}^{E}\right)K_I{}^{ME}+$$
$$\left.+\frac{3(a_5+2a_2)\left(2\lambda^2+13\lambda\mu+19\mu^2\right)+(a_5+\varepsilon)(\lambda+5\mu)(3\lambda+7\mu)}{2(a_5+2a_2)\left(3\lambda^2+18\lambda\mu+23\mu^2\right)+(a_5+\varepsilon)(\lambda+5\mu)(3\lambda+7\mu)}\left(K_{II}{}^{MC}+K_{II}{}^{ME}+K_{II}{}^{E}\right)K_{II}{}^{ME}\right\}+$$
$$+\varepsilon I^2 \tag{19}$$

Making use of formula (14), the criterion can be expressed as the dependence on the coefficients K_I^{MC}, K_I^{E}, K_{II}^{MC}, K_{II}^{E}, I

$$\gamma = \frac{1}{4\mu}\left\{\frac{\lambda+2\mu}{\lambda+\mu}\left[\left(K_I{}^{MC}\right)^2+\left(K_{II}{}^{MC}\right)^2\right]+\right.$$
$$-\frac{1}{\varepsilon-2a_2}\left[(a_5+2a_2)\frac{6\lambda^2+31\lambda\mu+33\mu^2}{(\lambda+5\mu)(3\lambda+7\mu)}+a_5+\varepsilon\right]K_I{}^{MC}K_I{}^{E}+$$
$$-2\frac{a_5+2a_2}{(\varepsilon-2a_2)^2}\frac{3\lambda^2+18\lambda\mu+23\mu^2}{(\lambda+5\mu)(3\lambda+7\mu)}\left[(a_5+2a_2)\frac{\mu(5\lambda+13\mu)}{(\lambda+5\mu)(3\lambda+7\mu)}-(a_5+\varepsilon)\right]\left(K_I{}^{E}\right)^2+$$
$$-\left[1+\frac{a_5+2a_2}{a_5+\varepsilon}\frac{12\lambda^2+57\lambda\mu+80\mu^2}{(\lambda+5\mu)(3\lambda+7\mu)}\right]K_{II}{}^{MC}K_{II}{}^{E}+$$
$$\left.+2\frac{a_5+2a_2}{a_5+\varepsilon}\frac{3\lambda^2+18\lambda\mu+23\mu^2}{(\lambda+5\mu)(3\lambda+7\mu)}\left[1+3\frac{a_5+2a_2}{a_5+\varepsilon}\frac{2\lambda^2+13\lambda\mu+19\mu^2}{(\lambda+5\mu)(3\lambda+7\mu)}\right]\left(K_{II}{}^{E}\right)^2\right\}+\varepsilon I^2 \tag{20}$$

The above formula seems more convenient for practical application than the previous one. Let us remind that the coefficients K_I^{E}, K_{II}^{E}, I are determined if the electric field in the dielectric is specified (Kurlandzka 1982, 1988, 1991).

Conclusions

It should be emphasized that the case of a crack of finite conductivity differs significantly from the case of a perfectly conducting crack. In the latter case the stress intensity coefficients K_I^{ME}, K_I^{E} are equal to zero. In the case considered they are not equal to zero and so electric stresses normal to the crack surface influence directly crack propagation.

Formula (20) looks more complicated than (19). However, the coefficients standing by the products and squares of the stress intensity coefficients K_I^{MC},

K_I^E, K_{II}^{MC}, K_{II}^E are in the case of a fixed material simply numerical coefficients. If the electric field in an dielectric is uniquely determined, the electric stress coefficients can be easily determined from formula (13) (KURLANDZKA 1982, 1988, 1991). The stress intensity coefficients K_I^{MC}, K_{II}^{MC} can be determined experimentally basing on strain measurement. However, it should be remembered that these coefficients are only a part of the stress intensity coefficient related to strains. Strain measurement leads to the determination of the coefficients K_I^M, K_{II}^M (11). The coefficients K_I^{MC}, K_{II}^{MC} should be determined from formula (14)

$$K_I^{MC} = K_I^M + \frac{(a_5+2a_2)(\lambda+\mu)(3\lambda+11\mu)+(a_5+\varepsilon)(\lambda+5\mu)(3\lambda+7\mu)}{(\varepsilon-2a_2)(\lambda+5\mu)(3\lambda+7\mu)} K_I^E,$$

$$K_{II}^{MC} = K_{II}^M + \frac{2\left(a_5+2a_2\right)\left(3\lambda^2+18\lambda\mu+23\mu^2\right)+(a_5+\varepsilon)(\lambda+5\mu)(3\lambda+7\mu)}{(a_5+\varepsilon)(\lambda+5\mu)(3\lambda+7\mu)} K_{II}^E.$$

Let us consider the criterion for dielectric material

$$0.9\ \text{PMN:0.1PT.}\quad \text{PMN: Pb}\ \ \text{Mg}_{\frac{1}{3}}\text{Nb}_{\frac{2}{3}}\text{O}_3,\ \ \text{PT: Pb Ti O}_3.$$

The material constants are YANG, SUO (1994):

Table 1

μ	λ	ε	a_2	a_5
44.44 GPa	48.15 GPa	66 404·10^{-12} MPa (cm/kV)^2	–0.016 MPa (cm/kV)^2	–0.135 MPa (cm/kV)^2

If the above values of the material constants are inserted into (20), the criterion assumes the form:

$$\gamma = 0{,}0083\left(\left(K_I^{MC}\right)^2+\left(K_{II}^{MC}\right)^2\right)+0{,}0584K_I^{MC}K_I^E+0{,}1160\left(K_I^E\right)^2+$$
$$-0{,}0230K_{II}^{MC}K_{II}^E-0{,}0320\left(K_{II}^E\right)^2+0{,}6640\times10^{-7}I^2.$$

Let us notice that the numerical coefficients standing by the products of the mechanical stress intensity coefficients and the electric stress intensity coefficients and by the squares of the electric stress intensity coefficients have higher values than those standing by the squares of the classical mechanical stress intensity coefficients. The minus sign by the term including K_{II}^E indicates that shear electric stresses counteract crack propagation. The direct influence of the electric field on crack propagation seems to be non-significant as compared with the influence of mechanical and electric stresses.

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