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ELECTROMAGNETIC FORCES IN POLARIZABLE, MAGNETIZABLE, CONDUCTING MEDIUM

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Key words: electromagnetic field, polarization and magnetization currents, polarization and magnetization charges, electromagnetic force, electromagnetic stress.

Abstract

The aim of the paper is an extension of the Dixon-Eringen (DIXON, ERINGEN 1965) model of the polarizable and magnetizable deformable dielectric to the media with conductivity. The author starts with the microscopic model assuming that besides the electric dipoles, which are taken into considerations in the Dixon-Eringen model, the magnetic dipoles according to the Chu concept of magnetization and free charges are present (FANO, CHU, ADLER 1960). Hence the macroscopic polarization, magnetization, conductive and convective currents densities, polarization and magnetization charges and currents densities are derived. Basing on these results the Lorentz force is obtained. In result the electromagnetic volume force and the electromagnetic stresses are received. In a sense they are an extension of the Dixon-Eringen model of electromagnetic interactions in the deformable dielectric to the polarizable, conducting media.

SIŁY ELEKTROMAGNETYCZNE W OŚRODKU Z POLARYZACJĄ, MAGNETYZACJĄ I PRZEWODNOŚCIĄ

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Słowa kluczowe: pole elektromagnetyczne, ładunki polaryzacji i magnetyzacji, prądy polaryzacji i magnetyzacji, siły elektromagnetyczne, naprężenia elektromagnetyczne.

Abstrakt

Celem pracy jest uogólnienie modelu dielektryka odkształcalnego Dixona-Eringena (DIXON, ERINGEN 1965) na ośrodek z przewodnością. Przyjęto koncepcję mikroskopowej struktury ośrodka jako składającej się z dipoli elektrycznych, dipoli magnetycznych, które zgodnie z hipotezą Chu są odpowiedzialne za magnetyzację (FANO, CHU, ADLER 1960), oraz ładunków swobodnych. Następnie na podstawie tego modelu określono makroskopowe wielkości, jak: polaryzacja, magnetyzacja, gęstości ładunków swobodnych, prądu przewodzenia i konwekcyjnego oraz gęstości ładunków i prądów polaryzacji i magnetyzacji. Na podstawie tych wielkości wyznaczono elektromagnetyczne siły objętościowe i naprężenia elektromagnetyczne. Uzyskane wyniki są uogólnieniem modelu dielektryka Dixona-Eringena, w którym nie występuje magnetyzacja na polaryzowalny ośrodek przewodzący.

Introduction

The aim of the paper is an extension of the Dixon-Eringen model of the ideal deformable dielectric to the model of the polarizable, magnetizable and conducting media.

R.C. Dixon and A.C. Eringen assumed a microscopic concept of the dielectric as the medium consisting of the electric dipoles and quadrupoles. Basing on the assumption, that on any charge of the dipole and of the quadrupole the Lorentz force acts, by means of an averaging procedure the electromagnetic force acting on the volume element of the medium was received. The assumption of the electric quadrupoles leads to the model of the polar dielectric. If the quadrupoles are neglected, the symmetric stress tensor is received.

The very same procedure, if it is assumed, that beside the electric dipoles there are free charges connected with the conductivity leads to the terms, which can not be expressed by means of the macroscopic electromagnetic quantities. However it turned out, that if the Chu concept (CHU, FANO, ADLER 1960) of polarization and magnetization charges and currents is applied, the microscopic approach after averaging procedures leads to the macroscopic electromagnetic quantities such as free charges and conductive and convective currents densities, polarization, magnetization, polarization and magnetization charge and currents densities. Basing on these quantities and the concept of the macroscopic Lorentz force the electromagnetic volume force and electromagnetic stresses are derived. The results are an extension of the Dixon-Eringen interactions in the polarizable deformable media to the interactions in polarizable deformable media with conductivity.

Charges and currents

Let us consider a volume dv of a medium. Let $\mathbf{x}$ denotes a position of its mass center. It is assumed, that in the volume there are the electric dipoles consisting of the couple of electric positive and negative charges q_α^+, q_α^-, $q_\alpha^- = -q_\alpha^+$, the magnetic dipoles of positive and negative magnetic charges μ_γ^+,

μ_γ^+, $\mu_\gamma^- = -\mu_\gamma^+$, and the free electric charges $q^f{}_\beta$. In the absence of an external electromagnetic field the positive and the negative charges of the electric and of the magnetic dipoles occupies the same place given respectively by means of the position vectors $\mathbf{x} + \zeta_\alpha$, $\mathbf{x} + \zeta_\gamma$, Under influence of the external electromagnetic field the charges move. The positive and the negative charges of the electric dipoles change their position respectively to $\mathbf{x} + \zeta_\alpha{}^+$, $\mathbf{x} + \zeta_\alpha{}^-$ while the positive and negative magnetic charges to $\mathbf{x} + \zeta_\gamma{}^+$, $\mathbf{x} + \zeta_\gamma{}^-$.

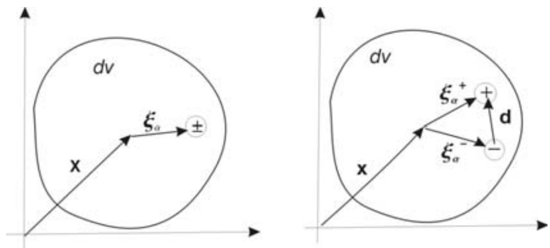

Fig. 1. The electric dipole before and after polarization

The electric and the magnetic moments of the dipoles are respectively

$$q_\alpha{}^+ \ (\boldsymbol{\zeta}_\alpha{}^+ - \boldsymbol{\zeta}_\alpha{}^-),\ \mu_\gamma{}^+(\boldsymbol{\zeta}_\gamma{}^+ - \boldsymbol{\zeta}_\gamma{}^-) \tag{1}$$

The motion of the free charges under influence of the external electric field generates an electric current

$$q_\beta{}^f(\dot{\mathbf{x}} + \dot{\boldsymbol{\zeta}}_\beta) \tag{2}$$

where $q_\beta{}^f\ \dot{\mathbf{x}}$ is the convection current and $q_\beta{}^f\dot{\zeta}_\beta$ is the conduction current (Eringen, Maugin 1989, Fano, Chu, Adler 1960).

The electric and magnetic moments cause polarization and magnetization of media. The electric and magnetic charge densities q, μ, the electric current density $\mathbf{j}$, the polarization and magnetization densities $\mathbf{p}$, $\mathbf{m}$ are defined by means of the formulas

$$\begin{gathered} q dv = \sum_\beta q_\beta{}^f,\ \mathbf{j} dv = \sum_\beta q_\beta{}^f(\dot{\mathbf{x}} + \dot{\boldsymbol{\zeta}}_\beta) = q\mathbf{v} dv + \hat{\mathbf{j}} dv \\ \mathbf{p} dv = \sum_\alpha q_\alpha{}^+(\boldsymbol{\zeta}_\alpha{}^+ - \boldsymbol{\zeta}_\alpha{}^-),\ \mathbf{m} dv = \sum_\gamma \mu_\gamma{}^+(\boldsymbol{\zeta}_\gamma{}^+ - \boldsymbol{\zeta}_\gamma{}^-) \end{gathered} \tag{3}$$

The term $q\mathbf{v}$ denotes the density of the convective current, $\hat{\mathbf{j}}$ is the conduction current density, $\mathbf{v}$ a velocity of the mass center of the volume element dv.

Under influence of the external electric field the positive charges of the electric dipoles move to the surface in the direction of the polarization vector $\mathbf{p}$, while the negative charges move in the direction opposite to the polarization vector. Thus on the surface da of the volume dv the external electric field generates the surface charge

$$\mathbf{p} \cdot \mathbf{n}\, da$$

where $\mathbf{n}$ denotes the normal unit vector external to the surface da.

Let q^p denotes the volume density of the polarization charges. The total number of the positive and of the negative charges of the electric dipoles before polarization and in the polarized body remains the same. In the polarized body the total number of these charges consists of the surface charges on the surface S

$$Q^p = \oint_S \mathbf{p} \cdot \mathbf{n}\, da = \oint_S \sigma^p\, da$$

where σ^p is the surface density of the polarization charges and from the charges in the volume V of the body

$$\int_V q^p\, dv$$

Here q^p denotes the volume density of polarization charges. Thus

$$\oint_S \mathbf{p} \cdot \mathbf{n}\, da + \oint_V q^p\, dv = 0$$

Hence by means of the Green theorem

$$\oint_S \mathbf{p} \cdot \mathbf{n}\, da = \int_V div = - \int_V q^p\, dv$$

the volume and surface densities of the polarization charges are respectively

$$q^{p} = -\,div\mathbf{p}, \quad \sigma^{p} = \mathbf{p}\cdot\mathbf{n} \tag{4}$$

An analogous analysis of the magnetization leads to the conclusion, that the surface and the volume densities of the magnetic charges are

$$\sigma^{m} = \mu_0\,\mathbf{m}\cdot\mathbf{n}, \quad q^{m} = -\,\mu_0 div\mathbf{m} \tag{5}$$

where μ_0 is the permeability of free space.

The electric and magnetic charges densities change in time. The changes are due to the dependence of the electromagnetic field on time and due to the deformation of the body. Taking into account the charge conservation

$$\oint_S \mathbf{j}\cdot\mathbf{n}da + \frac{d}{dt}\int_V q\,dv = 0$$

and (4), (5)

$$\int_V q^{p}\,dv = -\int_V div\mathbf{p}dv = -\oint_S \mathbf{p}\cdot\mathbf{n}\,da$$

$$\int_V q^{m}\,dv = -\int_V \mu_0\,div\mathbf{m}dv = -\oint_S \mu_0\,\mathbf{m}\cdot\mathbf{n}\,da$$

it results

$$\oint_S \mathbf{j}^{p}\cdot\mathbf{n}da + \frac{d}{dt}\oint_S \mathbf{p}\cdot\mathbf{n}\,da, \quad \oint_S \mathbf{j}^{m}\cdot\mathbf{n}da + \frac{d}{dt}\oint_S \mu_0\,\mathbf{m}\cdot\mathbf{n}\,da \tag{6}$$

where $\mathbf{j}^{p}$, $\mathbf{j}^{m}$ are the densities of the polarization and magnetization currents.

Let us remind, that the deformable body is considered, then

$$x^{k} = x^{k}\,(X^{K},\,t), \quad p^{l}(x^{k},\,t) = p^{l}[x^{k}(X^{K},\,t),t],\ n_l da = da_l$$

$$\dot{x}^{k} = \left.\frac{dx^{k}}{dt}\right|_{\mathbf{X}\ \mathrm{fixed}} = \frac{\partial x^{k}}{\partial t},\ \frac{dp^{l}}{dt} = \frac{\partial p^{l}}{\partial t} + p^{l}{}_{;\mathrm{k}}\,v^{k}$$

where **X** denotes the material coordinates, **x** the spatial coordinates of the material points of the body, $p^l{}_{;k}$ is the covariant derivative of p^l. For the time derivative of an integral of a field $\phi[x^k(X^K,t),t]$ over the material surface S the following dependence holds (ERINGEN 1962)

$$\frac{d}{dt}\int_S \phi\, da_k = \int_S \left[\left(\frac{\partial \phi}{\partial t} + \phi_{;m} v^m\right) da_k + \phi(-\, v^m{}_{;k}\, da_m + v^m{}_{;m}\, da_k)\right]$$

Hence the polarization and magnetization currents densities are

$$\mathbf{j}^{\mathbf{p}} = \frac{\partial \mathbf{p}}{\partial t} + \mathbf{v}\, div\, \mathbf{p} + rot(\mathbf{p} \times \mathbf{v}) = \overset{*}{\mathbf{p}}$$
$$\mathbf{j}^{\mathbf{m}} = \frac{\partial(\mu_0 \mathbf{m})}{\partial t} + \mathbf{v}\, div(\mu_0 \mathbf{m}) + rot(\mu_0 \mathbf{m} \times \mathbf{v}) = \mu_0 \overset{*}{\mathbf{m}} \tag{7}$$

The stars denote the convective derivatives (ERINGEN, MAUGIN 1989).

Electromagnetic forces

The electric free charges and currents, polarization and magnetic charges and currents defined in the previous section contribute to the forces acting in the body. The volume charge densities and the current densities contribute to the volume force acting on the volume element of the body

$$\mathbf{f} = \mathbf{f}^{\mathbf{e}} + \mathbf{f}^{m}$$
$$\mathbf{f}^{\mathbf{e}} = (q^f + q^p)(\mathbf{e} + \mathbf{v} \times \mathbf{b}) + (\hat{\mathbf{j}} + \mathbf{j}^p) \times \mathbf{b} \tag{8}$$
$$\mathbf{f}^{\mathbf{m}} = q^m(\mathbf{h} - \mathbf{v} \times \mathbf{d}) - \mathbf{j}^{\mathbf{m}} \times \mathbf{d}$$

The density of the electric free charge, polarization charge density, conduction current, and polarization current contribute to the electric part of the volume force $\mathbf{f}^{\mathbf{e}}$. The magnetic charges and the magnetic current contribute to the magnetic part $\mathbf{f}^{\mathbf{m}}$ of the volume force. In the above formulas **e**, **h** denote the electric and magnetic field intensity, **d**, **b** are the electric displacement and the magnetic induction. The last quantities can be expressed by means of the electric and magnetic intensities and polarization and magnetization (FANO, CHU, ADLER 1960, SUFFCZYŃSKI 1969).

$$\mathbf{d} = e_0\mathbf{e} + \mathbf{p}, \quad \mathbf{b} = \mu_0(\mathbf{h} + \mathbf{m}) \tag{9}$$

Inserting the last dependences and the formulas (4), (5), (7) into (8) one obtains the electric and magnetic parts of the volume force in the form

$$\begin{aligned} \mathbf{f^e} &= (q - div\mathbf{p})\mathbf{e} + [\hat{\mathbf{j}} + (q - div\mathbf{p})\mathbf{v} + \overset{*}{\mathbf{p}}] \times \mathbf{b} \\ \mathbf{f^m} &= -\mathbf{h}\,div\mathbf{m} - (\overset{*}{\mathbf{m}} - \mathbf{v}\,div\mathbf{m}) \times (e_0\mathbf{e} + \mathbf{p}) \end{aligned} \tag{10}$$

On the surface $\mathbf{n}da$ polarization and magnetic charges $\mathbf{p} \cdot \mathbf{n}da$, $\mathbf{m} \cdot \mathbf{n}da$ appear. Hence on the surface S of the body the electromagnetic force appears

$$\oint \mathbf{t}da = \oint \{\sigma^p(\mathbf{e} + \mathbf{v} \times \mathbf{b}) + \sigma^m\ (\mathbf{h} - \mathbf{v} \times \mathbf{d})\}\, da$$

Hence, it results

$$\begin{aligned} \oint_S t^k n_k da &= \oint_S [p^k(e_l + \varepsilon_{lmn} v^m b^n) + \mu_0 m^k(h_l - \varepsilon_{lmn} v^m d^n)] n_k da \\ &= \int_V [p^k(e_l + \varepsilon_{lmn} v^m b^n) + \mu_0 m^k(h_l - \varepsilon_{lmn} v^m d^n)]_{;k} dv \\ &= \int_V \tau^k{}_{l;k}\, dv \end{aligned}$$

Here $\tau^k{}_l$ denotes the electromagnetic stress tensor

$$\tau^{kl} = p^k e^l + \mu_0 m^k h^l + \varepsilon^{lmn} v_m(p^k b_n - m^k d_n) \tag{11}$$

Electromagnetic and mechanical field equations

To determine the electromagnetic field in the deformable medium let us start with the electromagnetic integral laws. For moving media it should be taken into considerations, that the effective electric and magnetic intensities are respectively

$$\mathbf{e} + \mathbf{v} \times \mathbf{b}, \quad \mathbf{h} - \mathbf{v} \times \mathbf{d}$$

Thus the integral laws can be written in the form:
The Faraday's law

$$\oint_C (e_k + \varepsilon_{klm} v^l b^m)\, dx^k = -\frac{d}{dt}\int_S b_k n^k da \tag{12}$$

The Ampére's law

$$\oint_C (h_k + \varepsilon_{klm} v^l d^m)\, dx^k = -\frac{d}{dt}\int_S d_k n^k da + \int_S (\hat{j}_k + v_k q) n^k da \tag{12}$$

The Gauss laws

$$\oint d_k n^k da = \int q\, dv, \quad \oint b_k n^k da = 0 \tag{14}$$

The time derivatives of the integrals over the material surfaces, Green and Stokes theorems lead to the local Maxwell equations in the form

$$rot(\mathbf{e} + \mathbf{v} \times \mathbf{b}) = -\overset{*}{\mathbf{b}}$$

$$rot(\mathbf{h} - \mathbf{v} \times \mathbf{d}) = \overset{*}{\mathbf{d}} + \hat{\mathbf{j}} + q\mathbf{v} \tag{15}$$

$$div\mathbf{d} = q, \quad div\mathbf{b} = 0$$

If (9), the polarization charge and current densities and the magnetization charge and current densities are inserted into the local equations (14), they can be written in the form

$$rot(\mathbf{e} + \mathbf{v} \times \mathbf{b}) = -\mu_0 \overset{*}{\mathbf{h}} - \mathbf{j}^{\mathbf{m}}$$

$$rot(\mathbf{h} - \mathbf{v} \times \mathbf{d}) = e_0\, \overset{*}{\mathbf{e}} + \mathbf{j}^p + \hat{\mathbf{j}} + q\mathbf{v} \tag{16}$$

$$e_0\, div\mathbf{e} = q^p + q, \quad \mu_0 div\mathbf{h} = q^m$$

Under assumption that the electromagnetic fields and the velocities of the points of media are finite, the boundary conditions on the surface S of the body resulting from (12)-(14) are of the form

$$[(\mathbf{e} + \mathbf{v} \times \mathbf{b}) - \mathbf{e}^v] \times \mathbf{n} = 0, \quad [(\mathbf{h} - \mathbf{v} \times \mathbf{d}) - \mathbf{h}^v] \times \mathbf{n} = 0$$

$$(\mathbf{d} - e_0\mathbf{e}^v) \cdot \mathbf{n} = 0, \quad (\mathbf{b} - \mu_0\mathbf{h}^v) \cdot \mathbf{n} = 0$$

where $\mathbf{e}^v$, $\mathbf{h}^v$ denote the electric and the magnetic intensities in free space. The form corresponding to (16) is

$$(\mathbf{e} - \mathbf{e}^v) \times \mathbf{n} + \mathbf{b}\,(\mathbf{v} \cdot \mathbf{n}) = 0, \quad (\mathbf{h} - \mathbf{h}^v) \times \mathbf{n} + \mathbf{d}(\mathbf{v} \cdot \mathbf{n}) = 0$$
$$e_0(\mathbf{e} - \mathbf{e}^v) \cdot \mathbf{n} = -\,\sigma^p, \quad \mu_0(\mathbf{h} - \mathbf{h}^v) \cdot n = -\,\sigma^m \tag{17}$$

Taking into considerations the electromagnetic force acting on the material volume dv of the body and the electromagnetic stress acting on its surface da the local form of the equations of motion are obtained in the form

$$(\sigma^{ij} + \tau^{ij})_{;i} + f^j + \rho F^i = \rho \frac{\partial^2 u^j}{\partial t^2}$$

where σ^{ij} is the mechanical part of the stress dependent on strains, $\rho\mathbf{F}$ is the mechanical body force, $\mathbf{u}$ the displacement vector.

If the stress tensor $\boldsymbol{\sigma} + \boldsymbol{\tau}$ is not symmetric, then the couple stress tensor appear

$$m^i{}_j = \varepsilon_{jlm}\,\mu^{ilm}$$

The couple stress satisfies the equation

$$\boldsymbol{\sigma}^{[jk]} + \boldsymbol{\tau}^{[jk]} + \mu^{ijk}{}_{;i} = 0$$

Here the square brackets denote the antisymmetric parts of the tensors σ, τ. The corresponding boundary conditions on the surface S are

$$(\boldsymbol{\sigma}_{ij} + \boldsymbol{\tau}_{ij})n^i = \tau^v{}_{ij}n^i + t_j$$

$$\mu_{ijk}\,n^i = 0$$

In the above formulas τ^v denotes the Maxwell stress tensor in free space

$$\tau^v{}_{ijk} = e_0 e^v{}_i e^v{}_j + \mu_0 h^v{}_i - \frac{1}{2}\,\delta_{ij}(e_0 e^v{}_k e^{vk} + \mu_0 h^v{}_k h^{vk})$$

The equations of the electromagnetic field and the equations of motion are coupled. The above model should be completed with the thermodynamic considerations leading to the constitutive equations for the polarization, magnetization, conductive current and mechanical parts of the stress tensor and the couple stress tensor. Such investigations will be the matter of further considerations.

Conclusions

In result of the investigations presented here the electromagnetic forces and stresses acting in the polarized and magnetized body with conductivity are derived. The approach is between (DIXON, ERINGEN 1965) and Chu formulation (FANO, CHU, ADLER 1960). The results differ from those derived by the authors reffered. Determination of the polarization and magnetization charge and current densities is a merit of the Chu approach. Owing to this the macroscopic Lorentz force can be derived. Because a deformable body is considered here, the polarization and magnetization currents density contain the additional terms if compared with the Chu results. The polarization and magnetization currents densities derived by Chu are (PENFIELD, HERMANN, HAUS 1967)

$$\mathbf{j}^p = \frac{\partial \mathbf{p}}{\partial \mathbf{t}} + rot(\mathbf{p} \times \mathbf{v}), \qquad \mathbf{j}^m = \frac{\partial \mathbf{m}}{\partial \mathbf{t}} + rot(\mathbf{m} \times \mathbf{v})$$

and do not contain the terms

$$\mathbf{v}\,div\mathbf{p}, \qquad \mu_0 \mathbf{v}\,div\mathbf{m}$$

present in (7). However the presented approach, differs from the Dixon--Eringen approach the electromagnetic volume force obtained by the author is the same as the electromagnetic volume force in (DIXON, ERINGEN 1965), if it is assumed that the media is not conducting and not magnetizable.

Moreover in Chu approach it was not taken into account, that the surface densities of polarization and magnetization charges generate the electromagnetic stresses. The electromagnetic stresses in Chu approach result from thermodynamic considerations and differs from these obtained by the author. Here again the electrical stresses in the Dixon-Eringen model are the same as derived by the author, if it is assumed, that the media is not magnetizable and the quadrupole effects are neglected. It should be noticed, that in the Dixon--Eringen approach the electromagnetic stresses are derived on a formal way as a term appearing in the form of divergence in the volume force resulting from the model. In the approach presented here the electromagnetic stress tensor result directly from the physical model.

In conclusion it should be emphasised, that the electromagnetic interactions derived by the author are an extension of the Dixon-Eringen interaction for the polarizable media to the polarizable and conducting media. It does not concerns magnetization as the still controversial Chu approach is completely different than commonly accepted.

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