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GEODETIC APPLICATION OF *R*-ESTIMATION – LEVELLING NETWORK EXAMPLES

Robert Duchnowski

Institute of Geodesy
University of Warmia and Mazury in Olsztyn

Key words: *R*-estimation, levelling network.

Abstract

The paper presents geodetic application of *R*-estimation. Proposed method concerns estimation of differences between parameters of two functional models. Such estimation seems very useful in many geodetic problems, e.g., when observations are disturbed with gross errors or when differences mentioned are displacements of geodetic network points. The paper presents general solutions of *R*-estimation but it focuses special attention on its geodetic applications. In particular, it concerns levelling networks. The estimation method proposed here can be applied, for example, to monitoring of reference mark stability. The paper presents particular solution of such problem, based on *R*-estimation principle. Theoretical derivations and analysis are illustrated with some numerical examples.

GEODEZYJNE ZASTOSOWANIA *R*-ESTYMACJI – PRZYKŁADY DLA SIECI NIWELACYJNYCH

Robert Duchnowski

Instytut Geodezji
Uniwersytet Warmińsko-Mazurski w Olsztynie

Słowa klucze: *R*-estymacja, sieć niwelacyjna.

Abstrakt

Praca przedstawia geodezyjne zastosowania *R*-estymacji. Proponowana metoda dotyczy estymacji różnic między parametrami dwóch modeli funkcjonalnych. Taka estymacja wydaje się bardzo użyteczna w rozwiązaniu wielu problemów geodezyjnych, np. kiedy obserwacje są zaburzone błędami grubymi lub kiedy wspomniane różnice są przemieszczeniami punktów osnowy geodezyjnej. Praca przedstawia ogóle rozwiązania *R*-estymacji, ale szczególną uwagę zwrócono na jej geodezyjne zastosowanie, przede wszystkim dotyczące sieci niwelacyjnych. Przedstawiony sposób estymacji może być na przykład stosowany do kontroli stabilności punktów nawiązania. Praca przedstawia sposób takiej kontroli oparty na ogólnych zasadach *R*-estymacji. Przedstawione wyprowadzenia i teoretyczne rozważania zilustrowano kilkoma przykładami numerycznymi.

Introduction

R-estimation is one of the fundamental ways of robust estimation, together with M-estimation and L-estimation (HUBER 1981). It is well know and often applied in many scientific researches, e.g., (MUKHERJEE, BAI 2001, ROUSSEEUW, VERBOVEN 2002). The main idea of R-estimation is to apply some statistical rank test for example the Wilcoxon one (HUBER 1981, FELTOVICH 2003) to estimate a shift between two samples. For example, from a geodetic point of view, those two samples can be interpreted as two sets of geodetic measurement results or results obtained in two different periods. Then the shift between the samples can be regarded as effect of geodetic point displacement or some non-random error of measurements. On the other hand, R-estimation can be also regarded as an estimation of differences between parameters of two different, functional models. Some geodetic applications of R-estimators were proposed in the paper (DUCHNOWSKI 2008).

R-estimation, due to its main principles, is robust against outliers and therefore can be an alternative for other robust methods of adjustment, e.g., methods of M-estimation class (e.g., HUBER 1981, XU 1989, YANG 1994, DUCHNOWSKI 2005). Robustness is an important property of R-estimation but it is not the only advantage. Another one is possibility to apply that estimation class to elimination of some non-random errors from observation sets (DUCHNOWSKI 2008). Presented R-estimation properties and applications make the method useful in solving of many geodetic problems. That paper describes application of R-estimation within a displacement measurement process. Particular solution concerns leveling networks however their assumptions and principles can also be extended for other kinds of geodetic networks.

Let us consider a leveling network established for deformation or displacement monitoring. Estimation of point heights that change in time is most natural application of R-estimation. Such estimation can be regarded as estimation of parameters of two different, functional models (models formulated for two different measurement periods). That application is illustrated later on with the first numerical example. It is also well known that stability of reference marks is essentially important in such problems. Several methods were elaborated to recognize stable reference framework, they are described in many papers and books (PRÓSZYNSKI, KWAŚNIAK 2006). However, those methods sometimes fail, e.g., if a set of measurement results is disturbed with a gross error. Thus it is advisable to have a chance to compare results obtained from different methods. The present paper proposes a new method of mark stability monitoring. That method is of course based on specially modified R-estimation. It is very important since its theoretic foundations are different from the classical ones (e.g., HUBER 1981, PRÓSZYNSKI, KWAŚNIAK

2006, DUCHNOWSKI 2008). Another advantage of the method is its robustness which facilitates monitoring of bench marks stability even when an observation set is disturbed with a gross error. The proposed method is also illustrated with numerical example.

R-estimation

Theoretical foundation and basic solutions

Let two independent samples $x_1, x_2, ., x_n$ and $y_1, y_2, .., y_m$ be realizations of random variables X and Y and let $F(x)$ and $G(y) = F(x - \Delta)$ be distributions of X and Y, respectively. Thus both distributions differ from each other in a quantity Δ, which can be regarded as a shift between two samples. The main tusk of R-estimation is to assess the shift Δ. The following test statistic can be formulated for such reason (HUBER 1981).

$$S_{n,m} = \frac{1}{m}\sum_{i=1}^{m} a(R_i) \tag{1}$$

where $a(i)$ are some given scores and R_i is a rank of x_i in the joint sample (containing all elements of two mentioned ones). Usage of a particular function $a(i)$ depends on the kind of assumed rank test (HUBER 1981, FELTOVICH 2003, NARAJDO, MCKEAN 1997). The R-estimation concept is to find such estimator $\hat{\Delta}_R$ of the shift Δ that makes $S_{n,m} = 0$ (or less strictly $S_{n,m} \approx 0$ considering some properties of $S_{n,m} \approx 0$, e.g., discontinuity) i.e., to search for such shift estimator that makes the test (1) unable to detect a difference in locations of two mentioned samples (or to make it least able for such detection).

Let us consider test (1). Scores $a(i)$ are usually generated using a function $J(t)$. For example

$$a(i) = J\left(\frac{i}{m+n+1}\right) \tag{2}$$

There are several other possibilities for scores $a(i)$, also for the same $J(t)$. They can be found in, e. g., HUBER (1981) or NARAJDO and MCKEAN (1997). If the Wilcoxon test is assumed, i. e., if

$$J(t) = t - \frac{1}{2} \tag{3}$$

then most often used functions $a(i)$ lead to the same test statistic (HUBER 1981). Assuming such form of $J(t)$ function, the following R-estimator of the shift can be formulated

$$\hat{\Delta}^R = \text{med}\,(y_i - x_j) \tag{4}$$

where med (∘) is a median operator and $1 \le i \le m,\ 1 \le j \le n$ (HUBER 1981).

Basing on the same assumptions another useful R-estimate can be derived. Consider only one sample $x_1, x_2, ..., x_n$ of the variable X. Now, let the expected value $E(X)$ of X be estimated. To solve the problem one can construct a "mirror image" sample, i.e., $2E(X) - x_1,\ 2E(X) - x_2,\ .,\ 2E(X) - x_n$, which can stand in for the sample $y_1, y_2, ..., y_m$. Then the following estimate (HUBER 1981)

$$\hat{E}^R(X) = \text{med}\left[\frac{1}{2}\,(x_i + x_j)\right] \tag{5}$$

$(1 \le i \le m + n,\ 1 \le j \le m + n)$, , makes $S_{n,m} = 0$ (or at least $S_{n,m} \approx 0$).

The R-estimates presented above are derived basing on the test statistic in Eq.(1) and the $J(t)$ function from Eq.(3). They can be regarded as some variants of the Hodges-Lehmann estimates, too (e.g., HUBER 1981). The estimator Eq.(5) should be used very carefully especially in geodetic applications where number of observations is not very high. It is due to properties of the Hodges-Lehmann estimators. It can be proved that in a very small sample. i.e., when $n < 5$, that estimate is not robust for outliers (ROUSEEUW, VERBOVEN 2002).

Adaptation for geodetic purposes

Both presented estimators can be useful in some geodetic elaborations (DUCHNOWSKI 2008). However, there are some limitations of their applications. The main limitation concerns assumptions of the method, i. e., that all sample elements are identically distributed. If that assumption is taken strictly then geodetic applications of R-estimation is limited to the case when only one quantity is measured and it is measured several times with the same accuracy. As for the second problem, it can be easy overcome if only measurement results are standardized or if one applies other rank test (FELTOVICH 2003). The way how to deal with the first limitation is presented in the paper (DUCHNOWSKI 2008) and it is now shown briefly.

Let $\mathbf{x} \in R^{n \times 1}$ be an observation vector. Generally, that vector consists of realizations $x_1, x_2, ..., x_n$ of different random variables, which makes

R-estimators (4) and (5) unable to use. Let us consider the classical functional model of a geodetic network in the form

$$\mathbf{v} = \mathbf{x} - \mathbf{AX} \tag{6}$$

where: $\mathbf{A} \in R^{n \times m}$ is a know, rectangular matrix, $\mathbf{X} \in R^{m \times 1}$ is a vector of unknown parameters and $\mathbf{v} \in R^{n \times 1}$ is a residual vector. Let $\tilde{\mathbf{X}}$ be a "true" or at least initial value of parameter $\mathbf{X}$ and let it be known. Then one can write

$$\tilde{\mathbf{v}} = \mathbf{x} - \mathbf{A}\tilde{\mathbf{X}} \tag{7}$$

and compute a new sample $\tilde{\mathbf{v}} = [\tilde{\mathrm{v}}_1, \tilde{\mathrm{v}}_2, ..., \tilde{\mathrm{v}}_n]^T$, (the notification is simplified for the sake of clarity; formally it is $\tilde{\mathrm{v}}_i = [\tilde{\mathbf{v}}]_i$),. Because all residuals $\tilde{\mathrm{v}}_1$, (the vector $\mathbf{v}$) can be regarded as identically distributed (assuming the same accuracy or residual standardization) then also $\tilde{\mathrm{v}}_i$, (the vector $\tilde{\mathbf{v}}$) can be (e.g., DIONE 1981). It means that $\tilde{\mathbf{v}}$ can be apply in the estimates (4) and (5). Since the true value of the parametric vector $\mathbf{X}$ stays unknown it is very important to find a "good" initial value $\tilde{\mathbf{X}}$. For example, it can be computed on the basis of measurement results x_1, x_2, ..., x_n or can be taken from the previous measurements or adjustments. The second way seems especially useful in deformation or displacement elaborations where current results are compared with the previous, control ones. Despite $\tilde{\mathbf{X}}$ is usually only an assessment of $\mathbf{X}$ and the assumption about identicalness of distributions of the vector $\tilde{\mathbf{v}}$, elements is not satisfied strictly, practically computed $\tilde{\mathbf{X}}$ should be good enough to neglect such nonconformity.

Particular Applications in Leveling Networks

Generally, R-estimation applications in leveling networks are the same like in other kinds of geodetic observational systems. R-estimates can be applied to, e.g., elimination of systematic errors or gross errors from observation sets. The method can be also used during classical least squares adjustment (LS method) for example in the R-LS method (DUCHNOWSKI 2008). This paper proposes applications of R-estimation in leveling networks created for deformation or displacement detection.

The first application is the most natural one and concerns vertical displacement estimation. Consider a leveling network created for such displacement detection. Such network should be measured at least two times resulting in two observation vectors $\mathbf{x}_1$ and $\mathbf{x}_2$. Basing on assumed functional model Eq. (6) one can write

$$\mathbf{v}_1 = \mathbf{x}_1 - \mathbf{A}\mathbf{X}_1 \tag{8}$$

and

$$\mathbf{v}_2 = \mathbf{x}_2 - \mathbf{A}\mathbf{X}_2 \tag{9}$$

for each epoch, respectively. Usually, heights of the network points are assumed as parameters in such functional models. Thus vectors $\mathbf{X}_1$ and $\mathbf{X}_2$ consist of heights of the same points but in different epochs. Let us now consider a single element $[\mathbf{X}_2]_k$ of the vector $\mathbf{X}_2$ that corresponds with the element $[\mathbf{X}_1]_k$. Then a difference $[\mathbf{X}_2]_k - [\mathbf{X}_1]_k$ can be regarded as a vertical displacement of the respective network point. The formula (5) can be applied to estimate such difference. Since the first measurements are control ones, the estimator $\hat{\mathbf{X}}_2$ (obtained using for example the LS method) can be regarded as a good initial value $\tilde{\mathbf{X}}_2$ for the second measurement set. Thus a sample $\tilde{\mathbf{v}}_2$, for the vector $\mathbf{v}_2$, can be easily created (according to Eq. (7)). One can now consider a limitation of the sample to only such elements that correspond with measurements concerning the parameter $[\mathbf{X}_2]_k$. Such sample can be denoted as $\tilde{\mathbf{v}}_2^k = [\tilde{v}_1^k, \tilde{v}_2^k, ..., \tilde{v}_l^k]^T$ (l is a number of observations concerning $[\mathbf{X}_2]_k$). Without losing generality, one can assume that all l elements of the vector $\mathbf{x}_2$ that were the basis for the computation of $\tilde{\mathbf{v}}_2^k$ are direct measurements of the kth point height. Thus $E([\mathbf{x}_2]_j) = E([\mathbf{X}_2]_j)$ (where $1 \leq j \leq l$). Since additionally $[\tilde{\mathbf{X}}_2]_k = [\hat{\mathbf{X}}_1]_k$ then

$$\hat{E}^R([\mathbf{X}_2]_k - [\mathbf{X}_1]_k) = \hat{E}^R([\mathbf{x}_2]_k - [\hat{\mathbf{X}}_1]_k) = \hat{E}^R(\tilde{v}^k)$$

can be regarded as an R-estimator of the vertical displacement of the respective network point. The value $\hat{E}^R(\tilde{v}^k)$ can be computed applying the formula (5)

$$\hat{E}^R(\tilde{v}^k) = \operatorname{med}\left[\frac{1}{2}\left([\tilde{\mathbf{v}}]_i^k + [\tilde{\mathbf{v}}]_j^k\right)\right] \tag{10}$$

In theory, such displacement estimator has all properties of R-estimators for example it is robust for outliers (as it was mentioned in the previous section the estimator is robust if $l \geq 5$).

The presented way of estimation is a good solution if reference marks are stable (or if only a few marks are not, taking into account robustness of the method). All reference marks cannot be guaranteed to be stable before such stability is controlled, i.e., before pointing out stable reference framework (what is the main task of monitoring process). Thus, the estimator Eq. (10) is not a good solution for stability monitoring problem. In such problem, it is also better to compare "raw" observation sets from two epochs than to use estimators $\hat{\mathbf{X}}_1$ from the first one (if $\tilde{\mathbf{X}} \neq \hat{\mathbf{X}}_1$ then one can avoid "bad" influences of observations that do not concern a particular reference mark). For such reasons, the estimator Eq. (4) seems to be the best application of R-estimation in a stability monitoring process. Thus one should create two samples $\tilde{\mathbf{v}}_1^k$ and $\tilde{\mathbf{v}}_2^k$ for all particular reference marks and for each epoch, respectively. The initial values $[\tilde{\mathbf{X}}]_k$ can be computed on the base of the first observation set taking into account only observations that correspond to the elements of $\tilde{\mathbf{v}}_1^k$. The shift Δ_k between the mentioned samples, which can be regarded as a vertical displacement of the reference mark, can be estimated applying the formula Eq. (4) written in the more convenient, following form

$$\hat{\Delta}_k^R = \text{med}\ ([\tilde{\mathbf{v}}_2^k]_i - [\tilde{\mathbf{v}}_1^k]_j) \qquad (11)$$

where $1 \leq i \leq l$, $1 \leq j \leq l$. Such computed shifts can be a basis for pointing out stabile reference framework. Reference marks can be regarded as stabile if respective estimated shift is inside of the interval assumed for random errors.

Numerical Examples

Example 1

Let us consider a leveling network that consists of three reference marks *A, B, C* where H_A = 1000 [m], H_B = 2000 [m], H_C = 3000 [m] and one unknown point 1 (it can be also regarded as a part of a bigger leveling network). Let three height differences (between each reference mark and the unknown point) be measured and let the measurement be carried out two times in two different epochs. Let

$$\mathbf{x}_1^T = [2.9990 \quad 1.9997 \quad 1.0013]_{[\text{m}]}$$

be a vector of the first measurement results (it is simulated under assumption that standard deviation of measurement σ = 0.001 [m] and results are normally distributed). Thus it can be computed $\hat{H}_{1I}^{LS}$. Let now

$$\mathbf{x}_2^T = [3.0140 \quad 2.0155 \quad 1.0158]_{[\mathrm{m}]}$$

be results of the second measurements (it is assumed that the vertical displacement of the point 1 is equal to 0.015 [m]; still $\sigma = 0.001$ [m]). The mentioned vertical displacement can be estimated applying Eq.(10) and one can obtained

$$\hat{E}^R([\mathbf{X}_2]_1 - [\mathbf{X}_1]^1) = \hat{E}^R(\tilde{\mathrm{v}}^1) = 0.0149 \text{ [m]}$$

The value presented here is very close to the true one 0.015 [m]. The displacement can be of course estimated applying the classical LS method. Then it can be calculated $\Delta\hat{H}_1^{LS} = 0.0151$ [m] which is also a good assessment of the true displacement (the difference between those two estimates cannot be big since the sample is very small and the observation sets are free of outliers (ROUSEEUW, VERBOVEN 2002, DUCHNOWSKI 2008)).

Let now the first observation in the vector $\mathbf{x}_2$ be disturbed with a gross error $g_1 = 0.02$ [m]. Then $\mathbf{x}_2^T = [3.0340 \quad 2.0155 \quad 1.0158]_{[\mathrm{m}]}$, and the presented displacement estimators can be computed as $\hat{E}^R(\tilde{\mathrm{v}}^1) = 0.0248$ [m] and $\Delta\hat{H}_1^{LS} = 0.0218$ [m], i.e., they both failed (the R-estimator failed because the sample is too small ($l = 3 < 5$); the LS estimator since that kind of estimates are not robust for outliers in general). However there is another way to estimate the displacement. One can consider the "raw" sets $\mathbf{x}_1$ and $\mathbf{x}_2$ and apply the R-estimator from Eq. (11). Then $\hat{\Delta}_1^R = 0.0165$ [m] which is much better assessment of the displacement (one can mention that for the original, not disturbed, observation set $\mathbf{x}_2$ it is $\hat{\Delta}_1^R = 0.0150$ [m]). What is more, if $g_1 = 0.10$ [m] then it is still $\hat{\Delta}_1^R = 0.0165$ [m] (in contrast, and $\hat{E}^R(\tilde{\mathrm{v}}^1) = 0.0648$ [m] and $\Delta\hat{H}_1^{LS} = 0.0484$ [m].

Example 2

Let us consider a leveling network established for detection of vertical displacements. Let the network contains four reference marks and let stability of those marks be controlled. Let all height differences between the reference marks be assumed as normally distributed with $\sigma = 0.001$ [m] and let they be measured twice. If the first and the fourth reference marks moved vertically and the displacement values are assumed as $\delta_1 = 0.025$ [m] and $\delta_4 = 0.005$ [m] respectively, then two measurement results can be simulated as follows.

$$\mathbf{x}_1 \begin{bmatrix} 0.9984 \\ 1.9994 \\ 0.9983 \\ -0.0001 \\ 1.0002 \\ 0.9999 \end{bmatrix}_{[\mathrm{m}]}, \mathbf{x}_2 = \begin{bmatrix} 0.9742 \\ 1.9716 \\ 0.9742 \\ 0.0008 \\ 0.9959 \\ 0.9965 \end{bmatrix}_{[\mathrm{m}]}, \text{ and } \mathbf{A} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

To find out the stabile reference framework one can eliminate those reference marks which moved vertically. Such movements (displacements) can be estimated with the R-estimator Eq. (11) (the vectors $\tilde{\mathbf{v}}_1^k$ and $\tilde{\mathbf{v}}_2^k$ (for $k = 1, 2, \ldots, 4$) can be created taking initial values $[\tilde{\mathbf{X}}]_k$ computed, for example, on the basis of the first observation set). Thus it can be computed that

$$\hat{\Delta}_1^R \begin{bmatrix} \hat{\Delta}_1^R \\ \hat{\Delta}_2^R \\ \hat{\Delta}^R \\ \hat{\Delta}_4^R \end{bmatrix} = \begin{bmatrix} 0.0251 \\ 0.0010 \\ -0.0008 \\ -0.0043 \end{bmatrix}$$

The obtained values of displacements are close to the assumed ones (as for the first and the fourth points). Taking into consideration the accuracy of the measurement ($\sigma = 0.001$ [m]) those two points cannot be regarded as stabile. It should be pointed out that also that time the R-estimator (11) shows its robustness (since two network points are assumed to be displaced vertically some results can be regarded as outliers).

The example presented above is rather simple but it illustrates the general idea of R-estimation application to monitoring of reference mark stability itself. The method is intended to be improved by the author.

Conclusions

R-estimation can be a useful statistical tool in some geodetic problems. It can be an alternative for classical approach to geodetic observation elaboration (e. g., for the LS method). Presented R-estimators Eq. (10) and Eq.(11) can be successfully applied to displacement detection herein leveling networks. As for the robustness against outliers the second estimate seems to be more useful (the first one can also be if only the observation number is high enough).

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References

CHEN Ch. 2001. *Rank estimating equations for random effects models.* Statistics and Probability Letters, 54, 5–12.

DIONNE L. 1981. *Efficient nonparametric estimators of parameters In the general Linear hypothesis.* The Annals of Statistics, 8(2): 457–460.

DUCHNOWSKI R. 2005. *Proposal of robust method of adjustment based on redundancy matrix application (in polish).* Technical Sciences, Supplement 2: 293–300.

DUCHNOWSKI R., WIŚNIEWSKI Z. 2006. *Attenuation Matrix in Robust, Free Adjustment.* Reports on Geodesy, 2(77): 299–306.

DUCHNOWSKI R. 2008. *R-estimation and its application to the LS adjustment.* Bollettino di Geodesia e Scienze Affini, 1: 21–30.

FELTOVICH N. 2003. *Nonparametric Tests of Differences in Medians: Comparison of the Wilcoxon-Mann-Whitney and Robust Rank-Order Tests.* Experimental Economics, 6: 273–297.

HUBER P.J. 1981. *Robust Statistics.* Willey, New York.

MUKHERJEE K., BAI Z.D. 2001. *R-estimation in Autoregression with Square-Integrable Score Function.* Journal of Multivariate Analysis, 81: 167–186.

NARANJO J.D., MCKEAN J.W. 1997. *Rank regression with estimated scores.* Statistics and Probability Letters, 33: 209–216.

PRÓSZYŃSKI W., KWAŚNIAK M. 2006. *Podstawy geodezyjnego wyznaczania przemieszczeń. Pojęcia i elementy metodyki.* Oficyna Wydawnicza Politechniki Warszawskiej.

ROUSSEEUW P. J., VERBOVEN S. 2002. *Robust estimation in very small samples.* Computation Statistics Data Analysis, 40: 741–758.

XU P. 1989. *On robust estimation with correlated observations.* Bulletin Geodesique, 63: 237–252.

YANG Y. 1994. *Robust estimation for dependent observations.* Manuscripta Geodaetica, 19: 10–17.