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ON CAPABILITY OF USING A FOUR POINT SPHERE PROBE TO FLOW VELOCITY MEASUREMENT

Zygmunt Wierciński
Institute of Fluid-Flow Machinery, PAS, Gdańsk
Faculty of Technical Sciences
University of Warmia and Mazury in Olsztyn

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Abstract

Measurement of the velocity is, next to the pressure measurement, among the most frequent measurements in mechanics of fluids and power engineering. That measurement is taken using, among others, the five-hole sphere probe. This paper presents the analysis of velocity measurement using a sphere probe showing that positioning of four measurement holes on the surface of the sphere is sufficient for measurement of the velocity direction and module. As a result we obtain a system of four nonlinear algebraic equations, which, however, not always possesses a solution. The paper presents four selected configurations of measurement holes on the sphere surface, three of which assure unequivocalness of problem solution. Additionally, the theoretical characteristic of five-hole sphere probe and its comparison with the experimentally obtained characteristic is presented.

O MOŻLIWOŚCI ZASTOSOWANIA CZTEROPUNKTOWEJ SONDY KULOWEJ DO POMIARU PRĘDKOŚCI

Zygmunt Wierciński
Instytut Maszyn Przepływowych PAN, Gdynia
Wydział Nauk Technicznych
Uniwersytet Warmińsko-Mazurski w Olsztynie

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S t r e s z c z e n i e

Pomiar wektora prędkości należy, oprócz pomiaru ciśnienia, do najczęściej wykonywanych w mechanice płynów i w eksploatacji maszyn i urządzeń energetycznych. Pomiaru tego dokonuje się m.in. za pomocą sondy kulowej pięciootworowej. W pracy przedstawiono analizę pomiaru wektora prędkości za pomocą sondy kulowej, wykazując, że do pomiaru kierunku i modułu wektora prędkości wystarcza umieszczenie czterech otworów pomiarowych na powierzchni kuli. W rezultacie otrzymujemy układ czterech równań nieliniowych, który jednak nie zawsze ma rozwiązania. Przedstawiono cztery wybrane konfiguracje czterech otworów pomiarowych na powierzchni kuli, z których trzy zapewniają jednoznaczność rozwiązania problemu. Ponadto zaprezentowano teoretyczną charakterystykę pięciootworowej sondy kulowej i jej porównanie z charakterystyka otrzymaną eksperymentalnie.

1. Introduction

Sphere probe is one of the most frequently used measurement instrument applied for measurement of velocity in a flow as it allows determining the velocity module and two angles in the coordinate system related to the probe. The measurement is taken through measurement of the distribution of pressure in selected points on the sphere surface, most frequently in five points distributed over the intersecting planes forming a cross on the sphere surface. The sphere probe surpasses the thermal anemometer probe in durability and simplicity of use. However, the sphere probe is most frequently used for measurement of the average velocity while the hot anemometer probe for measurement of velocity changing over time. If we additionally consider small dimensions, e.g. the diameter of 5 mm and possibility of applying pressure sensors of small size and fast dynamic reaction in measurement points on the surface, we can obtain the measurement tool allowing analysis of fast changing velocity fluctuations within a small measurement space. Lack of adequately small sensors offering low sensitivity threshold has made construction of such probes impossible so far.

Measurement of the required parameter using the least expensive means is one of the basic principles in the experimental technique. Five measurement hole sphere probes on the sphere surface are used as a valid standard of a sphere probe in the current measurement practice.

The objective of this paper is to show that assuming incompressibility of the flow measurement of the velocity using only four measurement holes on the sphere surface, i.e. no fewer than four measurement points is possible. Obviously, application of the sphere with five or more measurement holes (seven or nine) assures a better mapping of the velocity fields and offers a better accuracy of the measurement, nevertheless miniaturization of the probe is simpler in case of a lower number of pressure measurement points.

In this paper, on the basis of the potential flow around the sphere, it was shown that solution of a system of four nonlinear equations, which, however, do not have a single-unequivocal solution for every distribution of pressure measurement points on the sphere surface, would suffice to solve that measurement issue. The paper presents four configurations of four measurement points (square, rhombus, right triangle and equilateral triangle) where three of those configurations (rhombus, right triangle and equilateral triangle) assure unequivocalness of the solution of the nonlinear system of equations.

Additionally, the paper presents the theoretical characteristic of five-hole sphere probe and compares it to the characteristic obtained experimentally confirming their sufficient compatibility within the range of angles of $-10^{\circ} < \alpha < 10^{\circ}$ and $-10^{\circ} < \beta < 10^{\circ}$.

2. Distribution of pressure on the sphere surface as the base for velocity measurement

The potential flow of a homogenous flow around the sphere on the basis of which the determination of the distribution of velocity and pressure on the surface of the sphere is possible offers the theoretical base for measurement using the sphere probe. In other words, knowledge of the pressure distribution on the surface of the sphere is necessary for measurement of velocity using the sphere probe as measurement of velocity is the measurement of the difference of static pressures in a number, usually five, measurement points in the surface of the sphere. The most frequently found sphere probe is then the five-hole probe possessing five holes positioned on the lines forming the even-armed cross that cross at the right angle on the surface of the sphere.

The flow around the sphere is one of the axial symmetry problems and as a consequence its description requires knowledge of the sphere radius r, the momentary position of the flow stagnation point and the angle θ between the flow stagnation point S and point P selected at random on the surface of the sphere.

Investigating the flow around a sphere by a fluid possessing the velocity U and density ρ, the distribution of velocity around the sphere is expressed by the following formula (e.g. GRYBOŚ 1998):

$$u = \frac{\partial \Phi}{\partial R} = -U \cos\theta \left[1 - \left(\frac{r}{R}\right)^3\right] \qquad v = \frac{1}{R}\frac{\partial \Phi}{\partial R} = -U \sin\theta \left[1 + \frac{1}{2}\left(\frac{r}{R}\right)^3\right] \tag{1}$$

where Φ is the velocity potential.

On the sphere surface, for $R = r$ it will be:

$$u = 0 \quad v = \frac{3}{2} U \sin\theta \tag{2}$$

Applying the Bernoulli equation, we will determine the distribution of pressure in point P of our interest on the sphere surface according to the following formula (Gryboś 1998):

$$p_P = p_0 - \frac{1}{2}\rho U^2\left(1 - \frac{9}{4}\sin^2\theta\right) \tag{3}$$

where p_0 is the tangent pressure in the flow. As known from the experiment (Fage 1936, quoted after White 1974), the distribution of velocity around a sphere differs from the one given in the solution of flow around a sphere by a perfect fluid. The real distribution of velocity around a sphere is expressed by the following formula:

$$v = \frac{3}{2}U(\theta - 0.2914\theta^3 + 0.09873\theta^5 - 0.0001984\theta^7) \tag{4}$$

That formula is right for the following range of the angle θ, $0 < \theta < 1.48$ radian and the Reynolds numbers below $Re_r = 2 \cdot 10^5$ (sphere radius based Reynolds number). For that velocity distribution the velocity maximum is found for the angle $\theta = 1.291$ radian = 72°, while for the potential flow around a sphere that maximum is found for $\theta = \pi/2$ =90°, which means that the difference is significant. Additionally, in the real flow, the break-off point of the boundary layer on the sphere surface changes and depends on the status of the boundary layer on the sphere surface (laminar or turbulent) so it is a dependent of the Reynolds number.

By applying the expansion of the sinus function, it is possible to determine the difference between the theoretical potential and the real distribution of velocity on a sphere:

$$\Delta v = \frac{3}{2}U(0.124\theta^3 - 0.0904\theta^5 + 0.2818\theta^7 + ...) \tag{5}$$

Despite those significant differences, we will continue our reasoning treating the potential flow around a sphere as a certain model for the sphere probe.

3. Measurement probe in a flow

The flow velocity measurement problem will be analyzed in two coordinate systems. All of them will be related to the sphere probe, which will serve measurement of velocity, or rather measurement of pressure in a number of points on the surface of the sphere. To solve the velocity measurement problem a transformation must be formulated allowing the transition between those two coordinate systems.

The first coordinate system is related to the position of the velocity relative to the probe. That system was presented in Fig. 1. In that system two planes perpendicular to each other and cutting through the axis of the probe z were determined. The zy plane is the deviation plane while the *zx* plane is the pitch plane. The projections of the velocity on those two planes determine the deviation angle α and the pitch angle β. In such a coordinate system the calibration of the sphere probe is usually done. To achieve the situation where the direction of velocity matches the probe axis, the sphere probe should be rotated by angles (α, β) respectively. The five-hole sphere probe possesses pressure measurement points distributed appropriately symmetrically in the deviation and pitch planes while one of the holes is positioned on the axis of the probe.

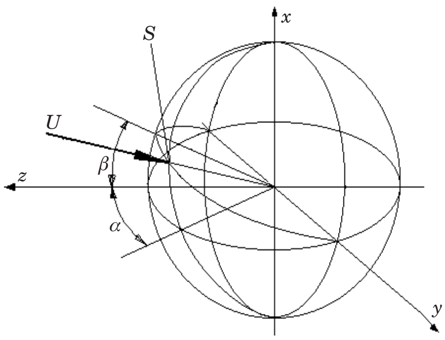

Fig. 1. System of coordinates for the sphere measurement probe and its positioning relative to velocity

In formula (3) for distribution of pressure in the surface of the sphere, we have an expression containing angle θ, as a consequence, an expression of angle θ depending on the coordinates of measurement point P and stagnation point S, Fig. 2.

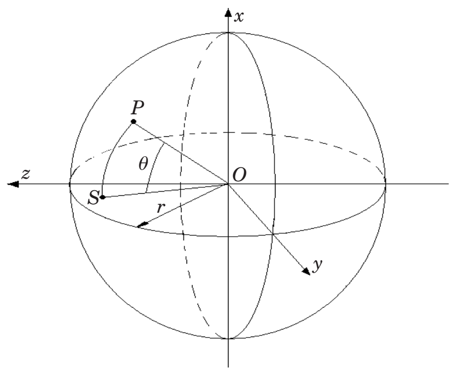

Fig. 2. Axial symmetry of flow around a sphere with the radius r: S – flow stagnation point, P – point in which the flow is analyzed (velocity and pressure), q – angle between points P and S

For that purpose we will try to determine the dependences between the system related to the measurement probe and the Cartesian system and after calculation of the linear distance between two points on the sphere we will try to determine the angle θ formed on that distance.

It can be shown that all points possessing the coordinate α or β are positioned on the ellipses corresponding to the following formulas created by projection of the great circle in the plane xy:

$$x^2 + \frac{y^2}{\sin^2 \alpha} = r^2 \qquad y^2 + \frac{x^2}{\sin^2 \beta} = r^2 \tag{6}$$

Those equations represent a simple system of equations with the unknowns x and y and the solutions of that system are provided by the following formulas:

$$x = \frac{r \cos\alpha \sin\beta}{\sqrt{1-\sin^2\alpha \sin^2\beta}} \qquad y = \frac{r \cos\beta \sin\alpha}{\sqrt{1-\sin^2\alpha \sin^2\beta}} \tag{7}$$

The z coordinate can be easily calculated from the Pythagorean theorem:

$$z = \frac{r \cos\alpha \cos\beta}{\sqrt{1-\sin^2\alpha \sin^2\beta}} \tag{8}$$

Next, searching for the value θ of the angle between two points on the surface of a sphere, we will use two formulas: for the distance between two points 1 and 2 and for the central angle formed at the distance of those two points:

$$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 \tag{9}$$

$$d = 2r \sin\frac{\theta}{2} \tag{10}$$

Inserting equations (7), (8) and (9) into (10) and using the identity:

$$\cos\theta = 1 - 2\sin^2\frac{\theta}{2} \tag{11}$$

after simple, but quite lengthy transformations, we obtained the expression:

$$\cos\theta = \frac{\cos(\alpha_1 - \alpha_2)\cos(\beta_1 - \beta_2) - \sin\alpha_1 \sin\alpha_2 \sin\beta_1 \sin\beta_2}{\sqrt{1-\sin^2\alpha_1 \sin^2\beta_1}\sqrt{1-\sin^2\alpha_2 \sin^2\beta_2}} \tag{12}$$

It is clear that in the system of Cartesian coordinates the calculations may be rather complicated and, possibly, even impossible. That is why it is necessary to apply the spherical coordinates system much more suitable for description of the measurement taken using a sphere probe.

The second coordinate system is permanently linked to the measurement probe axes ($r = 1$, φ, ψ), Fig. 3. In that system the position coordinates of pressure measurement points on the surface of the sphere ($r = 1$, φ_i, ψ_i) are defined. Also in that system the temporary position of the flow stagnation point on the sphere probe will be searched for; as a consequence the theoretical pressure field on the sphere surface is determined in relation to that temporary stagnation point.

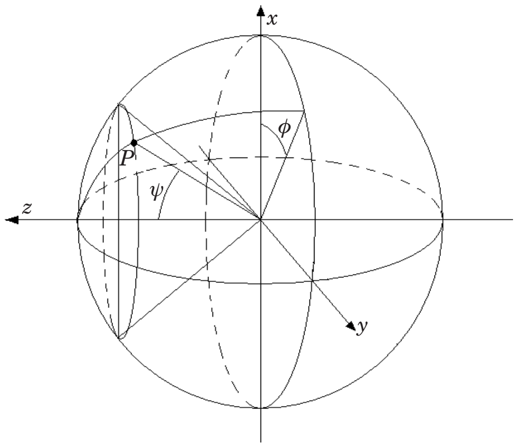

Fig. 3. Spherical coordinate system (r, φ, ψ) for sphere probe, z – probe axis

Below, in formulas (13) the generally applied spherical coordinates are presented with a minor deviation as instead of the symbol θ the symbol ψ was used leaving the θ for the coordinate of the temporary axially symmetrical flow around the sphere, i.e. the flow around independent off the ϕ coordinate was retained.

$$x = r \sin\psi \cos\varphi \quad y = r \sin\psi \sin\varphi \quad z = r \cos\psi \tag{13}$$

As in practice the α, β system is applied, it is necessary to find a transition between those two coordinate systems. For that purpose, the Cartesian coordinates from the expression (7 and 8) are treated as corresponding to the spherical coordinates (13). Following simple transformations we obtain two formulas linking angles ϕ and ψ with angles α and β respectively:

$$\mathrm{tg}\,\varphi = \frac{\mathrm{tg}\,\alpha}{\mathrm{tg}\,\beta} \qquad \mathrm{tg}^2\,\psi = \mathrm{tg}^2\,\alpha + \mathrm{tg}^2\,\beta \tag{14}$$

and the other way round, angles α and β with angles ϕ and ψ:

$$\operatorname{tg}\alpha = \operatorname{tg}\psi \sin\varphi, \qquad \operatorname{tg}\beta = \operatorname{tg}\psi \cos\varphi \tag{15}$$

As in the formula for the distribution of pressure on the surface of a sphere the θ is present, we must find an appropriate expression linking angle θ with angles ϕ_1 and ψ_1 as well as ϕ_2 and ψ_2 of the two points on which angle θ is spread.

Inputting (13) and (10) into (9) and again applying the identity (1), following a series of transformations, we obtain the expression:

$$\cos\theta = \sin\psi_1 \sin\psi_2 \cos(\varphi_1 - \varphi_2) + \cos\psi_1 \cos\psi_2 \tag{16}$$

It can be noticed easily that for $\varphi_1 - \varphi_2 = 90^\circ$, also (16) assumes the form of Pythagorean theorem for the spherical triangle built on the angle (section) θ and two angles (sections) ψ_1 and ψ_2, (STIEPANOW 1960). The spherical triangle based on the section θ and the vertex in the point where the axis cuts through the sphere surface. For $\varphi_1 = \varphi_2$ also (16) will have the form of the expression for the cosines of the difference between the angles ψ_1 and ψ_2, and for $\varphi_1 - \varphi_2 = 180^\circ$ – cosine of the sum of angles ψ_1 and ψ_2, as section θ is positioned on the circumference of the great circle cutting through the z axis. As a consequence, the above relations confirm that formula (16) is true.

If expression (14) (of course appropriately transformed) is input into formula (16), then we will obtain the expression identical with the formula (12), which also confirms that the calculations made so far are true.

Expression (16) already has a much simple form than expression (12) and it will be used for further calculations.

As we have obtained a simple expression for cos θ, it is also worth transforming the equation (3) to the form containing the cosine function instead of the sine. We will use the simple trigonometric one here:

$$p = p_0 + \frac{1}{2}\rho U^2 \left(-\frac{5}{4} + \frac{9}{4}\cos^2\theta\right) \tag{17}$$

To facilitate further calculations, the equation describing the pressure distribution on the sphere surface can be presented in a more compact form:

$$p = A + B\cos^2\theta, \quad \text{gdzie} \quad A = p_0 - \frac{5}{8}\rho U^2 \quad B = \frac{9}{8}\rho U^2 \tag{18}$$

In the calculations for the probe we will apply a system of indices different from that used so far. The index will represent the value (of, e.g. pressure, angle θ, coordinates) for the measurement point marked by that index. The values without indices apply to the stagnation point.

As a consequence, the pressure in the measurement point i will be expressed by the formula:

$$p_i = A + B\cos^2\theta_i \tag{19}$$

where:

$$\cos\theta_i = \sin\psi\sin\psi_i\cos(\varphi - \varphi_i) + \cos\psi\cos\psi_i \tag{20}$$

and as a consequence the pressure may be expressed as:

$$p_i = A + B(\sin\psi\sin\psi_i\cos(\varphi - \varphi_i) + \cos\psi\cos\psi_i)^2 \tag{21}$$

In that equation we have four unknowns: A, B, ψ and φ. As a consequence, it is enough to measure the pressure in four points on sphere surface, i.e. for φ_i ψ_i , i = 1...4 only to determine the velocity in a flow. In that way we obtain a system of four nonlinear equations with four unknowns the conditions of solution unequivocalness should be tested. To find those unknowns we must solve the system of four independent equations, i.e. input into equation (21) four different sets of coordinates, that is measure pressure in four different points of the sphere.

4. Five-hole sphere probe

Before we get to the solution of the problem of velocity measurement using a probe with four measurement points, however, we will solve the problem of velocity measurement using the five-hole sphere probe as use of such a probe is a standard for velocity measurements. The coordinates of the five holes on a standard sphere probe surface are presented in Table 1 while the graphic presentation of positioning the measurement points in a five-hole sphere probe is shown in Fig. 4.

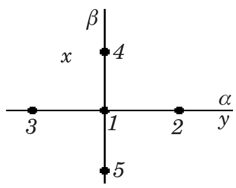

Fig. 4. Schematic presentation of positioning of the holes on a five-hole sphere probe

That numbering of points is probably most frequently found in the literature (BERNARD, HORODKO 1990).

The equations of pressures for those five points are presented in formula (21) according to the general formulas for pressure on sphere surface:

$$p_i = A + B(\sin\psi\sin\psi_i\cos(\varphi - \varphi_i) + \cos\psi\cos\psi_i)^2 \qquad i = 1...5 \tag{22}$$

After inputting appropriate values from Table 1, we obtain the following system of equations:

$$\begin{aligned} p_1 &= A + B\cos^2\psi \\ p_2 &= A + \frac{B}{2}(\sin\psi\sin\varphi + \cos\psi)^2 \\ p_3 &= A + \frac{B}{2}(-\sin\psi\sin\varphi + \cos\psi)^2 \\ p_4 &= A + \frac{B}{2}(\sin\psi\cos\varphi + \cos\psi)^2 \\ p_5 &= A + \frac{B}{2}(-\sin\psi\cos\varphi + \cos\psi)^2 \end{aligned} \tag{23}$$

Table 1

Coordinates of points on the surface of the five-hole standard sphere probe

Point No	ψ	φ	α	β
1	0	-	0	0
2	$\pi/4$	$\pi/2$	$\pi/4$	0
3	$\pi/4$	$-\pi/2$	$-\pi/4$	0
4	$\pi/4$	0	0	$\pi/4$
5	$\pi/4$	π	0	$-\pi/4$

Solving that system also through elimination of A and B, and particularly creating appropriate differences of pressures it is easy to write the formulas for the deviation angle α and pitch angle β respectively:

$$2\,\mathrm{tg}\,2\alpha = \frac{p_2 - p_3}{p_1 - \frac{p_2 + p_3}{2}} \qquad 2\,\mathrm{tg}\,2\beta = \frac{p_4 - p_5}{p_1 - \frac{p_4 + p_5}{2}} \tag{24}$$

The right sides of those formulas are used for calibration of the sphere probe and identified as coefficients KALPHA and KBETA, e.g. POENSGEN (1989), and SMOLNY et al. (1994). As we can see for an ideal standard sphere probe, coefficients KALPHA and KBETA can be calculated using very simple trigonometric dependences. Using equations (23) and knowing the values of angles α and β, the values of coefficients A and B can be calculated from the following formulas:

$$B = \frac{p_2 - p_3}{2\sin\psi\cos\psi\sin\varphi} = \frac{p_4 - p_5}{2\sin\psi\cos\psi\cos\varphi} \tag{25}$$

$$A = p_1 - B\cos^2\psi = p_1 - \frac{p_2 - p_3}{2\,\mathrm{tg}\,\alpha} = p_1 - \frac{p_4 - p_5}{2\,\mathrm{tg}\,\beta} \tag{26}$$

from where it is easy to calculate the dynamic pressure and the static pressure in the flow.

Recapitulating the results of chapter 4 it can be stated that the perfect standard five-hole sphere probe does not require any calibration to determine the velocity in a perfect fluid. It is worth highlighting once again that in case of standard sphere probe we deal with a system of five nonlinear algebraic equations with four unknowns and thanks to the double symmetry of measurement points the solution of that system of equations is really very simple.

In Fig. 5 the experimentally obtained characteristic of the sphere probe is presented (SMOLNY et al. 1994), while Fig. 6 presents the characteristic of the perfect five-hole sphere probe. The comparison of those characteristics shows imperfections in production of the real probe such as skew of the measurement holes in relation to the reference plains and lack of probe positioning in z axis.

In the theoretical characteristic the lack of uniformity in Kbeta increases depending on the linear increase of the angle is observed as a consequence of the dependence of Kbeta from the tangent of the double angle of deviation and pitch.

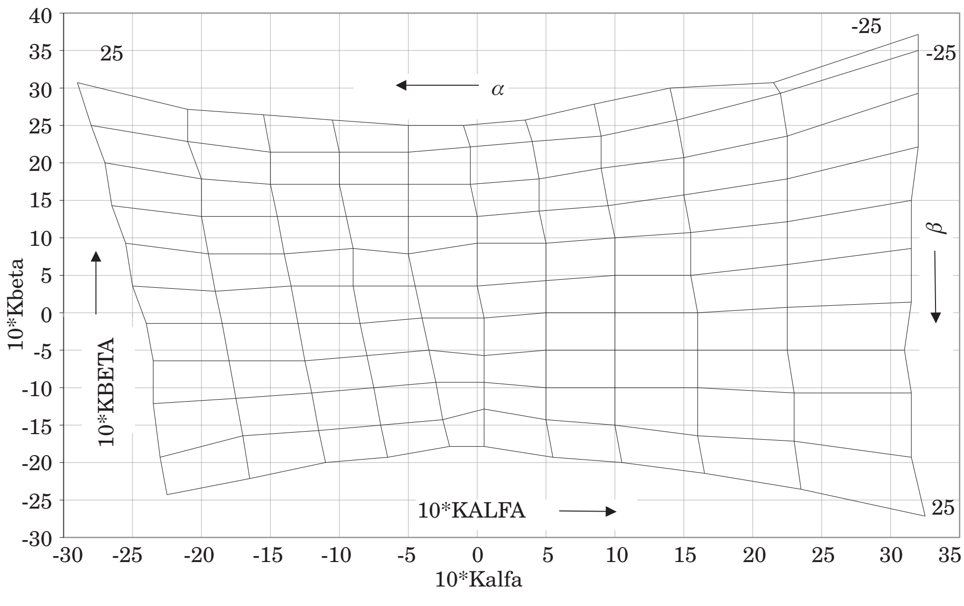

Fig. 5. Characteristic of a five-hole sphere probe (SMOLNY et al. 1994)

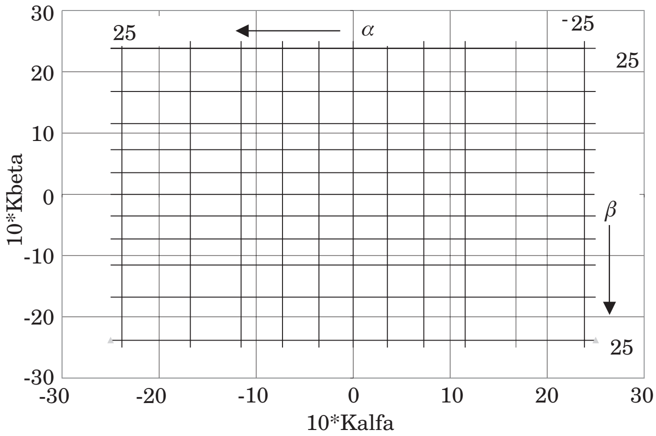

Fig. 6. Theoretical characteristic of a five-hole sphere probe

Nevertheless, it can generally be stated that both those characteristics are sufficiently accurate for the range of angles $-10^{\circ} < \alpha < 10^{\circ}$ and $-10^{\circ} < \beta < 10^{\circ}$. Additionally the probe from the example in the paper by SMOLNY et al. (1994) is basically a hemisphere and as a consequence its shape differs significantly from the perfect sphere. In short, it seems that much can still be done to bring the experimental characteristic closer to the theoretical one.

5. Four-hole sphere probe

5.1. Solution of the nonlinear algebraic equations system

As mentioned above, four measurement points should suffice to determine all the unknowns so a system of four algebraic equations with four unknowns should be developed.

An infinite number of setups of four measurement points could be proposed allowing calculation of the wanted unknowns. However, we will limit ourselves to four measurement setups that seem to be the simplest, Fig. 7:

a) a symmetrical rectangular setup is created by removing the central point from the five-hole probe setup,
b) a symmetrical rhomboidal setup similar to the one in point a) but with the measurement points positioned at two different distances from the center point,
c) the right triangle developed by removing one of the side measurement points from the five-hole probe,
d) the equilateral triangle with the measurement points at vertexes and center of the triangle.

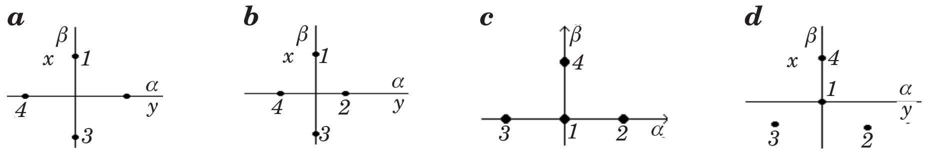

Fig. 7. Setup of four measurement points on sphere surface

At the beginning of our considerations we should verify whether the system of equations developed using the measurement of pressure in four points on the sphere surface has a solution.

$$p_i = A + B\left(\sin\psi \sin\psi_i \cos(\varphi - \varphi_i) + \cos\psi \cos\psi_i\right)^2 \qquad i = 1...4 \tag{27}$$

It is necessary to specify a certain criterion determining whether the equations in the system are independent.

The equations (27) in general are not linear equations so we may not apply a simple criterion for linear independence to them. Additionally it may be found that those equations are independent only in some points of phase space (A, B, φ, ψ), while in the other points of that space they may be dependent and then the system of equations in such cases would be impossible to provide one unequivocal solution.

To test independence of the equations we will write the equations (27) in the following format:

$$f_i(A, B, \varphi, \psi) = 0 \tag{28}$$

where:

$$f_i(A, B, \varphi, \psi) = -p_i + A + B\left(\sin\psi \sin\psi_i \cos(\varphi - \varphi_i) + \cos\psi \cos\psi_i\right)^2 \tag{29}$$

The condition of independence of the equations (27) is that the determinant will not zero (SCHWETLICK 1979):

$$W = \det\left[\frac{\partial f_i}{\partial x_j}\right] \neq 0 \tag{30}$$

where x_j represents consecutive unknowns A, B, φ and ψ. As it is easy to verify there is a fourth order determinant.

$$\begin{aligned}
&\frac{\partial f_i}{\partial A} = 1 \qquad \frac{\partial f_i}{\partial B} = b_i = \left(\sin\psi \sin\psi_i \cos(\varphi - \varphi_i) + \cos\psi \cos\psi_i\right)^2 \\
&\frac{\partial f_i}{\partial \varphi} = c_i = -2B \sin\psi \sin\psi_i \sin(\varphi - \varphi_i)\left[\sin\psi \sin\psi_i \cos(\varphi - \varphi_i) + \cos\psi \cos\psi_i\right] \\
&\frac{\partial f_i}{\partial \psi} = d_i = B\left[\sin 2\psi\left(\sin^2\psi_i \cos^2(\varphi - \varphi_i) - \cos^2\psi_i\right) + \sin 2\psi_i \cos(\varphi - \varphi_i)\cos 2\psi\right]
\end{aligned} \tag{31}$$

As all values of the first column of the determinant are equal to one, in calculating it the matrix can be made smaller by deducting raw 1 from the consecutive rows 2, 3 and 4 obtaining for calculation the third order determinant. For the general case, it is not easy to calculate the value of determinant W, and as a consequence we will limit ourselves to calculating it for the individual cases.

5.2. The square setup

We will analyze the case from point a) in Fig. 7. For the general solution of that setup we assume that $\psi_i = \psi_i = \psi_c$ i.e. the fixed value of angle ψ_i, while the angles $\varphi_1 = 0$, $\varphi_2 = \pi/2$, $\varphi_3 = \pi$ i $\varphi_4 = 3\pi/2$.

However, already for the first condition, i.e. $\psi_i = \psi_j = \psi_c$ and any values of φ_i *it* can be shown that the determinant W is equal to zero as:

$$b_{i1} = \sin^2\psi \sin^2\psi_c \sin(2\varphi - (\varphi_i - \varphi_1))\sin(\varphi_i - \varphi_1) \tag{32}$$

where the terms b_{i1} were formed through deductions of the terms of the first series of the determinant from the others, i.e. $i = 2$, 3 and 4. Using the formulas for the sum and difference of cosines it can be shown easily that all three terms $b_{i1} = 0$, when

$$\varphi = 0.5(\varphi_i - \varphi_1) \tag{33}$$

and that is enough for the value of the determinant W to be equal to zero. And that means that if the flow stagnation point is situated on the diagonal of the angles of two measurement points φ_i and φ_j, the system of equations has no solution indifferent of the value of angle ψ_c.

On the other hand the value of the determinant W for a four-hole probe developed from a five-hole probe by removing the center point can be calculated more easily as: $\psi_i = \psi_j = \psi_c = \pi/4$, and the angles $\psi_1 = 0$, $\psi_2 = \pi/2$, $\psi_3 = \pi$ and $\psi 4 = 3\pi/2$ can be calculated from the following formula:

$$W = 2B^2 \sin^3\psi \cos\psi\left(\sin^2\varphi - \cos^2\varphi\right) \tag{34}$$

It can be seen that in this particular case the determinant W zeroes (the system has no unequivocal solution) in the following circumstances:

a) $B = 0$ – in case there is no flow, and as a consequences angles φ and ψ cannot be determined (a trivial case),
b) $\psi = 0$ – it is impossible to determine the angle φ unequivocally, the stagnation point matches the probe axis,
c) $\psi = 90^\circ$ – that case will not be considered as formula (3) is not satisfied for such large angles that go beyond the sphere probe measurement range,
d) $\mathrm{tg}\varphi = \pm 1$ i.e. $\varphi = (\varphi_i - \varphi_j)/2 = \pi/4$ – it is a special case of condition (33). As a consequence such a setup of four measurement points positioned in corners of the square is unsuitable for practical application.

Similar result may be obtained while searching for a solution of the system of equations for such a rectangular (square) setup of measurement points as solving that system of equations we will obtain:

$$
\begin{aligned}
p_1 &= A + \frac{1}{2} B\left(\sin^2 \psi \cos^2 \varphi + 2\sin\psi \cos\psi \cos\varphi + \cos^2 \psi\right) \\
p_2 &= A + \frac{1}{2} B\left(\sin^2 \psi \sin^2 \varphi + 2\sin\psi \cos\psi \sin\varphi + \cos^2 \psi\right) \\
p_3 &= A + \frac{1}{2} B\left(\sin^2 \psi \cos^2 \varphi - 2\sin\psi \cos\psi \cos\varphi + \cos^2 \psi\right) \\
p_4 &= A + \frac{1}{2} B\left(\sin^2 \psi \sin^2 \varphi - 2\sin\psi \cos\psi \sin\varphi + \cos^2 \psi\right)
\end{aligned}
\tag{35}
$$

Deducting from sides the terms we will obtain:

$$p_1 - p_3 = 2B \sin\psi \cos\psi \cos\varphi \tag{36}$$

$$p_2 - p_4 = 2B \sin\psi \cos\psi \sin\varphi \tag{37}$$

where:

$$\operatorname{tg}\varphi = \frac{p_2 - p_4}{p_1 - p_3} \tag{38}$$

and:

$$\operatorname{tg}\psi = 2\left(p_1 - p_2 + p_3 - p_4\right)\frac{\sqrt{\left(p_1 - p_3\right)^2 + \left(p_2 - p_4\right)^2}}{\left(p_1 - p_3\right)^2 - \left(p_4 - p_2\right)^2} \tag{39}$$

It can be seen that the expression (39) makes no sense if $(p_1 - p_3) = \pm (p_2 - p_4)$. Then we deal with an indeterminate form type 0/0. On the basis of the equation (38) we may determine that it occurs for $\operatorname{tg}\varphi = \pm 1$. In that case measurement points 1 and 2 are at the same distance from the stagnation point and at the same time points 3 and 4 are also at the same distance from the stagnation point (or in pairs : points 1 and 4 and points 2 and 3 respectively). As a consequence in the system of equations (35) only two equations are independent and we lack three unknowns: A, B and ψ. Of course in that situation the system possesses no unequivocal solution. As a consequence a different setup of measurement points or an additional fifth equation is required.

5.3. Rhomboidal setup

Let us review the other proposed four point setups because, as stated in the previous section the setup of four points in a square does not provide the unequivocal solution.

The coordinates of measurement points for the rhomboidal setup are: $\varphi_1 = 0$, $\varphi_2 = 90^\circ$, $\varphi_3 = 180^\circ$, $\varphi_4 = 270^\circ$, $\psi_1 = \psi_3$, $\psi_2 = \psi_4$, $\psi_1 \neq \psi_2$ respectively and we determine precisely the values of ψ_1 and ψ_2.

Similar to the square setup of the measurement points we will first determine the value of determinant W for the rhomboidal setup. After quite arduous calculations that determinant can be presented using the following formula:

$$W = -B^2 \sin 2\psi \sin 2\psi_1 \sin 2\psi_2$$
$$\left[\cos 2\varphi \sin^2 \psi - \cos 2\psi_1 (1 - \sin^2 \varphi \sin^2 \psi) + \cos 2\psi_2 (1 - \cos^2 \varphi \sin^2 \psi)\right] \quad (40)$$

It can be shown easily that applying $\psi_1 = \psi_2 = \pi/4$ we will obtain the value of the determinant W the same as in case of the square setup according to the formula (34).

To find the points for which the determinant value is zero and the system of equations will not have solutions, it is enough to make the expression in square brackets equal to zero.

Assuming that $\psi_1 = \pi/4$ we can analyze the existing solutions of the system of equations in relation to the position of the second point when we, of course, assume that $\psi_2 < \pi/4$. Then the analysis of condition (40) can be brought to analysis of the following condition (41):

$$\sin^2 \psi = \frac{\cos 2\psi_2}{1 - \cos^2 \varphi (2 - \cos 2\psi_2)} \quad (41)$$

Of course the value of $\sin^2\psi$ must be lower than one (or at least equal to it) and that condition is satisfied when the following condition is satisfied:

$$\cos^2 \varphi \leq= \frac{1 - \cos 2\psi_2}{2 - \cos 2\psi_2} \quad (42)$$

From that condition the limit value of the angle φ can be determined for which $\psi < \pi/2$, i.e. we are searching for satisfaction of that condition for the front part of the probe. To immediately determine the minimum range of the angle ψ, where the system of equation cannot be solved, it is enough to assume that φ equals 0 or π and then

$$\sin^2 \psi = \cos 2\psi_2 \quad (43)$$

Assuming next $\psi_2 = \pi/6$ we see from formula (43), that the indeterminacy point of the system of equations is positioned in point $\varphi = 0$ and $\psi = \pm\pi/4$, i.e. in the position of measurement points number 1 and 3, and as a consequence, that point is positioned at the edge of the measurement area. If, on the other hand, we assume $\psi_2 = \pi/8$, the indeterminacy point for the system of equations is also positioned at $\varphi = 0$ and $\psi = \pm\, 57{,}23$ o, i.e. outside the area of velocity measurement using the sphere probe. As a consequence, the probe with rhomboidal setup and pressure measurement points for $\psi_1 = \pi/4$ and $\psi_2 = \pi/8$ fully satisfies the conditions of equivocal velocity measurement.

After determining the area of determinacy for solutions of the system of equations we may further search for the solution to the problem of velocity measurement using four-hole sphere probe with a rhomboidal setup.

Solving the system of equations for that setup of measurement points we obtain:

$$\frac{p_2 - p_4}{p_1 - p_3} = \operatorname{tg}\varphi \frac{\sin 2\psi_2}{\sin 2\psi_1} \tag{44}$$

From that formula it is easy to calculate the value of angle φ, if only the numerator and denominator are different from zero. For the case where $p_1 - p_3 = 0$ and/or $p_2 - p_4 = 0$, the problem simplifies as we will have $\alpha = 0$ and/or $\beta = 0$ respectively. Now we should only find the second equation allowing calculation of angle ψ, if angle φ is known. This can be achieved in a number of ways:

$$\frac{p_2 - p_4}{p_{1/3} - \frac{p_2 + p_4}{2}} = \frac{2\operatorname{tg}\psi \sin\varphi \sin 2\psi_2}{\operatorname{tg}^2\psi \sin^2\varphi - \operatorname{tg}^2\psi \cos^2\varphi \pm \operatorname{tg}\psi \cos\varphi \sin 2\psi_1 + \cos^2\psi_1 - \cos^2\psi_2} \tag{45}$$

and

$$\frac{p_1 - p_3}{p_{2/4} - \frac{p_1 + p_3}{2}} = \frac{2\operatorname{tg}\psi \cos\varphi \sin 2\psi_1}{\operatorname{tg}^2\psi \sin^2\varphi - \operatorname{tg}^2\psi \cos^2\varphi \pm \operatorname{tg}\psi \cos\varphi \sin 2\psi_1 + \cos^2\psi_1 - \cos^2\psi_2} \tag{46}$$

Notation $p_{1/3}$ should be understood as alternative use of p_1 or p_3 and then in the expression in the denominator the minus or plus should be used as appropriate.

Slightly simpler formulas are obtained (in the denominator the component with the tangent is out) from the further two formulas:

$$\frac{p_2 - p_4}{\frac{p_1 + p_3}{2} - \frac{p_2 + p_4}{2}} = \frac{2\operatorname{tg}\psi \sin\varphi \sin 2\psi_2}{\operatorname{tg}^2\psi \sin^2\varphi - \operatorname{tg}^2\beta \cos^2\varphi + \cos^2\psi_1 - \cos^2\psi_2} \tag{47}$$

$$\frac{p_1 - p_3}{\frac{p_1 + p_3}{2} - \frac{p_2 + p_4}{2}} = \frac{2\,\mathrm{tg}\,\psi \cos\varphi \sin 2\psi_1}{\mathrm{tg}^2\,\psi \sin^2\varphi - \mathrm{tg}^2\,\beta \cos^2\varphi + \cos^2\psi_1 - \cos^2\psi_2} \qquad (48)$$

The procedure of solving the problem of velocity measurement using a four-hole sphere probe with rhomboidal setup of measurement points is as follows: first, using the transformed equation (44) the value of the tangent of angle φ is calculated

$$\mathrm{tg}\,\varphi = \frac{p_2 - p_4}{p_1 - p_3} \frac{\sin 2\psi_1}{\sin 2\psi_2} \qquad (49)$$

then we determine the values of sin φ and cos φ, and in particular the so-called quarter of angle φ i.e. the signs of those function from analysis of the signs of differences $p_1 - p_3$ i $p_2 - p_4$, in which the following table can be helpful:

Table 2

Signs of the angles α, β and φ depending on the pressure differences signs

$p_1 - p_3$	β	$p_2 - p_4$	α	φ
0	0	0	0	–
>0	>0	>0	>0	$0 \ldots \pi/2$
<0	<0	>0	>0	$\pi/2 \ldots \pi$
<0	<0	<0	<0	$\pi \ldots 3\pi/2$
>0	>0	<0	<0	$3\pi/2 \ldots \pi$

For the case $p_1 - p_3 = 0$ and $p_2 - p_4 = 0$ angle φ is indeterminate while angle $\psi = 0$. In the following step, using one of the equations (45) to (48) the angle ψ value is calculated from a simple quadratic equation, which may be solved by a standard method. The root of the solution should be selected in a way satisfying the condition $0 < \psi < \pi/2$.

Further, the value of coefficient *A* can be calculated easily from one of the following formulas:

$$B = \frac{p_1 - p_3}{2\,\mathrm{tg}\,\psi \cos\varphi \sin 2\psi_1} = \frac{p_1 - p_3}{2\,\mathrm{tg}\,\beta \sin 2\psi_1} = \frac{p_2 - p_4}{2\,\mathrm{tg}\,\psi \sin\varphi \sin 2\psi_2} = \frac{p_2 - p_4}{2\,\mathrm{tg}\,\alpha \sin 2\psi_2} \qquad (50)$$

and the *B* value from the formula:

$$A = \frac{p_1 + p_3}{2} - \frac{p_2 - p_4}{\mathrm{tg}\,\psi \sin\varphi \sin 2\psi_2} \left(\sin^2\psi \cos^2\varphi \sin^2\psi_1 + \cos^2\psi \cos^2\psi_1\right) \qquad (51)$$

and further it is simple to calculate the value of velocity module U when medium density (or dynamic pressure $0.5\rho U^2$) and the static pressure p_0 are known.

5.4. The right triangle setup

In the analysis of the right triangle setup we will limit the discussion to a simple setup where angles $\psi_2 = \psi_3 = \psi_4 = 45^\circ$, which slightly simplifies the calculations. The coordinates of the measurement points then will be as follows:

$$\psi_1 = 0,\ \ \psi_2 = \psi_3 = \psi_4 = 45^\circ,\ \ \ \varphi_2 = 0,\ \ \varphi_3 = 90^\circ,\ \ \varphi_4 = -90^\circ.$$

Developing the system of equations for that setup of measurement points we obtain the following formula (as points 1, 3 and 4 are positioned on one plane):

$$2\,\mathrm{tg}\,2\alpha = \frac{p_3 - p_4}{p_1 - \dfrac{p_3 + p_4}{2}} \tag{52}$$

Using that formula we can determine angle α and the only thing that is left is to determine angle β. In this case we also have two options:

$$\frac{p_3 - p_4}{p_2 - \dfrac{p_3 + p_4}{2}} = \frac{4\,\mathrm{tg}\,\alpha}{\mathrm{tg}^2\beta - \mathrm{tg}^2\alpha + 2\,\mathrm{tg}\,\beta} \tag{53}$$

to calculate tgβ.

Calculation of angle β is also possible from another formula:

$$\frac{p_1 - \dfrac{p_3 + p_4}{2}}{p_2 - \dfrac{p_3 + p_4}{2}} = \frac{1 - \mathrm{tg}^2\alpha}{\mathrm{tg}^2\beta - \mathrm{tg}^2\alpha + 2\,\mathrm{tg}\,\beta} \tag{54}$$

Those are simple quadratic equations that can be solved easily by standard methods. Further calculations of A and B values are easy.

5.5. Equilateral triangle setup

In this case the coordinates of the measurement points are: $\psi_1 = 0$, $\psi_2 = \psi_3 = \psi_4 = 45^\circ$ $\varphi_1 = 0$, $\varphi_2 = 120^\circ$, $\varphi_3 = -120^\circ$ respectively. Solving the system of equations for that setup we obtain:

$$\frac{p_3 - p_1}{p_2 - p_1} = \frac{\frac{1}{4}\left(\operatorname{tg}\alpha - \sqrt{3}\operatorname{tg}\beta\right)^2 - \left(\operatorname{tg}\alpha - \sqrt{3}\operatorname{tg}\beta\right) - 1}{\operatorname{tg}^2\alpha + 2\operatorname{tg}\alpha - 1} \tag{55}$$

and:

$$\frac{p_4 - p_1}{p_2 - p_1} = \frac{\frac{1}{4}\left(\operatorname{tg}\alpha + \sqrt{3}\operatorname{tg}\beta\right)^2 - \left(\operatorname{tg}\alpha + \sqrt{3}\operatorname{tg}\beta\right) - 1}{\operatorname{tg}^2\alpha + 2\operatorname{tg}\alpha - 1} \tag{56}$$

The obtained system of equations is simplified to the form:

This is a system of two quadratic equations with unknowns tgα and tgβ. After appropriate transformations and elimination of one unknown, i.e. tgβ, we obtain the following quartic equation:

$$\begin{aligned} &\operatorname{tg}^4\alpha(p_3 - p_4)^2 + \operatorname{tg}^3\alpha\left[4(p_3 - p_4)^2 - (2p_3 + 2p_4 - 3p_1 - p_2)(p_2 - p_1)\right] + \\ &+ 2\operatorname{tg}^2\alpha(p_3 - p_4)^2 + \operatorname{tg}\alpha\left[(6p_1 - 10p_2 - 2p_3 - 2p_4)(p_2 - p_1) - 4(p_3 - p_4)^2\right] + \\ &+ \left[(p_3 - p_4)^2 - 4(p_3 + p_4 - 2p_2)(p_2 - p_1)\right] = 0 \end{aligned} \tag{57}$$

This is a quartic equation because of tgα. We can develop it using Cartesian-Euler substitutions or by applying the method given by NEUMARK (1965). Next, as usually, the values of A and B must be calculated from the series of equations followed by calculation of dynamic and static pressures.

6. Conclusion

The paper presents the analysis of velocity measurement using a sphere probe. The five-hole sphere probe is a standard solution. The new idea was to use a system of spherical coordinates in the calculations. We succeeded in finding a transition between that coordinate system and the coordinate system using the deviation angle α and pitch angle β applied as a standard. We also found a criterion according to which the practical suitability of four-hole sphere probe can be assessed as it allows verifying whether the system of equations describing the pressures in measurement points is a system of independent equations in every case. The paper shows that velocity measurement is possible by applying a four-hole sphere probe. However, positioning of measurement points in vertexes of a square (i.e. in the setup obtained by removing the center point from the five-hole measurement probe) gives a system of equations that is indeterminate. On the other hand, positioning of measurement points in vortexes of a rhombus provides one-unequivocal solution to the problem of velocity measurement. In this case we obtain a simple equation for angle φ while the value of angle ψ should be calculated form a quadratic equation. Similarly, in case of measurement points setup in the form of a right triangle we obtain a simple equation for angle α (or β) and the second angle β (or α) should be calculated from a quadratic equation. In case of the measurement setup in the form of equilateral triangle the situation is slightly more complex as a system of quadratic equations must be solved, which in consequence means solving a single quartic equation.

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