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FRACTURE OF ELASTIC DIELECTRICS IN AN ELECTRIC FIELD

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Abstract

Results of investigations on the influence of an electric field on fracture of a deformable dielectric are presented in the paper. Basing on the Toupin's model of electro-elastic interactions, modified by the author, a fracture criterion has been formulated, the behaviour of electro-elastic fields in the vicinity of the crack tip has been analysed and a generalization of the Irwin criterion in the case of electro-elastic fields has been derived.

PĘKANIE DIELEKTRYKÓW SPRĘŻYSTYCH POD WPŁYWEM POLA ELEKTRYCZNEGO

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Słowa kluczowe: pękanie, pole elektryczne, dielektryk.

Streszczenie

Przedstawiono wyniki badań teoretycznych nad wpływem pola elektrycznego na pękanie dielektryków odształcalnych. Opierając się na modelu oddziaływań elektro-sprężystych Toupina, zmodyfikowanym przez autorkę, sformułowano kryterium pękania, podano wyniki analizy zachowania się pól elektro-sprężystych w otoczeniu naroża szczeliny oraz uogólniono kryterium Irwina na przypadek pól elektro-sprężystych.

Introduction

The fact that an electric field generates strains in a dielectric material has been known for a long time. If the dielectric can deform without restraints, there are no stresses inside the material. However if there are some constraints, this phenomenon causes the generation of stresses. A magnitude of the stresses generated by an electric field depends on the properties of the material and on the field intensity.

For years the stresses generated by an electric field in elastic materials were not taken into account, because there was no good theoretical description of the media. The first theory of electromagnetic interactions with the elastic media was formulated in the late fifties of 20th century (CHADWICK 1956, TOUPIN 1956). Then a significant progress took place in this field.

Although it was known from experiments that a strong electric field generates significant stresses in some materials, the influence of an electromagnetic field on fracture was not taken into account before 1975. In six papers were published in this field, four by KURLANDZKA (1975) and two by KUDRIAVCEV, PARTON, RAKITIN (1975). In the next years an interest in this field increased. Apart from the papers by Polish and Russian authors, papers by Japanese and American authors appeared. Due to the development of new technologies, particularly in control systems, the interest in this field still increases.

Some results of theoretical investigations by the author on electromagnetic fracture are presented in the paper. Here only a static theory, concerning the influence of an electric field on crack propagation, will be presented for a simplified model. Detailed considerations concerning a full nonlinear problem can be found in KURLANDZKA (1998). Results concerning a vacuum and a perfectly conducting crack form a full theory. The problem of a crack of finite conductivity is not closed, and investigations will be continued.

Generalized Griffith criterion for the elastic dielectric

A crucial point of each fracture theory is the formulation of a fracture criterion. In a pure mechanical case there are various criteria, including energetic criteria based on an energy balance of a body containing a crack. It seems that in the case of electro-elastic interactions only an energetic approach to fracture is a proper one. It results from the fact that the crack propagates not only due to the stresses caused by strains of the material, but also due to the electric stresses and energy of an electric field. In the considerations presented here a basis for the formulation of the fracture criterion is the Griffith's statement, according to which some energy is required to create a new surface of the crack. In the present paper the energy which can be used for the creation of a new crack surface is called a fracture energy.

To formulate the criterion one has to refer to a consistent and full theory of elastic dielectrics. The model used here has been developed by Toupin (1956). The version used here was modified to be more suitable in applications to investigations of the influence of a strong electric field on fracture Kurlandzka (1998).

The model of the dielectric consists of the following system of coupled equations

1) equilibrium equations:

$$\left(\sigma_k{}^l + t_k{}^l + \tau_k{}^l\right)_{;l} + \rho f_k = 0 \quad in \;\; V_d \tag{1}$$

2) equation for the electrostatic field in a dielectric

$$D_{;l} = 0 \quad in \;\; V_d, \qquad D^l = d^l - \varepsilon^{kl}\phi_{;k} \tag{2}$$

3) equation for the electrostatic field in a vacuum or in a conductor of a finite conductivity

$$e_0\phi_{;}{}^k{}_k + q = 0 \quad in \;\; V_v, \qquad \breve{\phi}_{;k}{}^k = 0 \;\; in \;\; V_c \tag{3}$$

Here V_d, V_v, V_c denote the domains occupied by the dielectric, a vacuum and a conductor of a finite conductivity. $\boldsymbol{\sigma}$, $\mathbf{t}$, $\boldsymbol{\tau}$ are mechanical, electroelastic and electric parts of the stress tensor respectively, $\rho\mathbf{f}$ is mass force, ρ is mass density, ϕ, $\breve{\phi}$ are electric potentials in the dielectric and in the outside domain, $\mathbf{d}$ is dielectric displacement generated by strains, ε, e_o are dielectric permittivity of the dielectric and a vacuum, q is electric charge density. The electric field intensity is $E_i = -\phi_{;i}$.

The above system of equations is supplemented by constitutive relations, which by assumption of small strains are of the form

$$\sigma_k{}^l = \rho\frac{\partial\Sigma\left(e_{ij}\right)}{\partial e^k{}_l}, \quad t_k{}^l = \rho\frac{\partial\eta\left(e_{ij}, E_m\right)}{\partial e^k{}_l}, \quad d^l = \rho\frac{\partial\eta\left(e_{ij}, E_m\right)}{\partial E_l} \tag{4}$$

$$\tau_k{}^l = \varepsilon^{ml}\phi_{;m}\phi_{;k} - \frac{1}{2}\varepsilon^{mn}\phi_{;m}\phi_{;n}\delta_k{}^l \tag{5}$$

where Σ is elastic energy which depends on the strains $e_{kl} = \frac{1}{2}\left(u_{k;l} + u_{l;k}\right)$, $\mathbf{u}$ is the displacement vector, h is energy of coupling of a mechanical and electric field being a function of the strains and the electric intensity $E_k = -\phi_{;k}$.

With the equilibrium equations the stress boundary conditions on the boundary of the dielectric are connected

$$\left(\sigma_k{}^l + t_k{}^l + \tau_k{}^l\right) n^k = \breve{\tau}_k{}^l n^k + T^l \ on\ S_d \tag{6}$$

where S_d is a boundary of the dielectric, **n** is a unit vector normal and external to the domain of the dielectric. $\breve{\tau}_k{}^l$ denotes electric stress in the outside domain, **T** is a surface mechanical load.

The electric boundary conditions depend on the electric properties of the media outside the dielectric. They have the form:

1) on the common boundary of the dielectric and vacuum

$$\left(d_k - \varepsilon_k{}^l \phi_{;l}\right) n^k = e_0 \breve{\phi}_{;k} n^k, \quad \phi = \breve{\phi} \tag{7}$$

2) on the boundary of the dielectric with a perfect conductor

$$\phi = 0 \tag{8}$$

3) on the boundary of the dielectric and a conductor of conductivity σ

$$\phi = \breve{\phi}, \quad \breve{\phi}_{;k} n^k = 0.$$

Let us consider an element of a dielectric of volume V_d bounded by a surface S_d and containing a segment of a crack of a surface S_c. A plane problem is considered here.

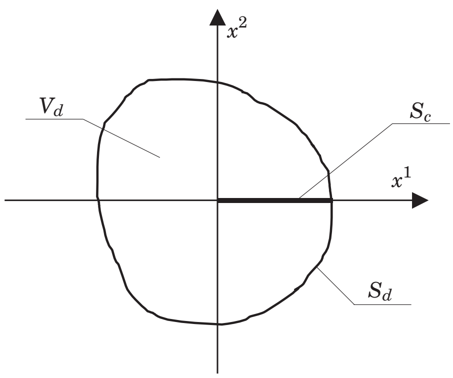

Fig. 1. Element of a dielectric containing a crack segment

The internal energy of the considered element of a dielectric is a sum of elastic energy, the energy of electro-elastic coupling and electrostatic energy

$$\int_{V_d}\left[(\rho\Sigma+\rho\eta)+\frac{1}{2}\varepsilon^{ik}\phi_{;ik}\right]dv.$$

If the surface of the crack increases, then a certain energy equal to a difference of a change of work of an external load and of a change of the internal energy due to the crack increment is used for the creation of a new surface δl.

The state of the dielectric is described by displacements u of the particles of the dielectric and the electrostatic potential ϕ. Variations of these quantities due to δl are

$$\delta u^k=\frac{\partial u^k}{\partial l}\delta l,\ \ \delta\phi=\frac{\partial\phi}{\partial l}\delta l.$$

Change of the work of external forces is equal to the work of the stresses on the displacement variations, and of the surface electric charge on the electrostatic potential variation on the surface of the dielectric S_d, on the crack surface S_c, and of the mass forces on the displacement variations in the volume of the dielectric V_d

$$\delta W=\int_{S_d}\left[(\sigma_{kl}+t_{kl}+\tau_{kl})n^l\delta u^k+q\,\delta\phi\right]ds+\int_{S_c}\left(\breve{\tau}_{kl}n^l\delta u^k+\breve{q}\,\delta\breve{\phi}\right)ds+\int_{V_d}\rho f_k\delta u^k dv.$$

Denoting by $\delta\Gamma$ the variation of the fracture energy due to the crack increment dl, the fracture criterion can be written in the form

$$\delta\Gamma=\int_{S_d}\left[(\sigma_{kl}+t_{kl}+\tau_{kl})n^l\delta u^k+q\,\delta\phi\right]ds+\int_{S_c}\left(\breve{\tau}_{kl}n^l\delta u^k+\breve{q}\,\delta\breve{\phi}\right)ds+\int_{V_d}\rho f_k\delta u^k dv$$
$$-\delta\int_{V_d}\left(\rho\Sigma+\rho\eta+\frac{1}{2}\varepsilon^{ik}\phi_{;i}\phi_k\right)dv.$$

Taking into account equations (1)–(7), the fact that the integrands can be singular, applying the Green-Gauss theorem and assuming that the increment of the crack is along the axis x^1 the energy G, which is supplied by the external generalized load and can be used for the creation of a new crack surface, is given by the formula:

$$G=-\frac{1}{2}\lim_{\xi\to 0}\int_{S_\xi}\Big\{\Big[(\sigma^{kl}+t^{kl}+\tau^{kl})u_{k;1}+\left(d^l-\varepsilon^{kl}\phi_{;k}\right)\phi_{;1}\Big]n_l$$
$$-\left[\rho(\Sigma+\eta)-\frac{1}{2}\varepsilon^{ik}\phi_{;i}\phi_{;k}\right]n_1\Big\}ds \tag{9}$$

Here S_ξ is a cylindrical surface of radius ξ surrounding the crack edge.

The right side of the expression determines the energy which is supplied to the dielectric by external load and can be used for the creation of the unit crack length.

The generalized Griffith criterion can be now formulated as follows:

If

$$-\frac{1}{2}\lim_{\xi\to 0}\int_{S_\xi}\left\{\left[\left(\sigma^{kl}+t^{kl}+\tau^{kl}\right)u_{k;1}+\left(d^l-\varepsilon^{kl}\phi_{;k}\right)\phi_{;1}\right]n_l - \left[\rho(\Sigma+\eta)-\frac{1}{2}\varepsilon^{ik}\phi_{;i}\phi_{;k}\right]n_1\right\}ds=\gamma \tag{10}$$

where γ is a material parameter called surface energy, then the crack is in a critical state and even a very small increment of the energy determined by a limit value of the integral on the left side of equality (10) can cause a rapid growth in its length.

The above criterion is a generalization of the Griffith criterion in the case of coupled electro-elastic fields. The integral form of the criterion is not convenient in practical applications. It can be simplified, if the behaviour of the functions present in the integrand in a vicinity of the crack tip is known. To this aim a local solution of the problem for a dielectric containing a crack should be developed.

Behaviour of electro-elastic fields in the vicinity of the crack tip

Criterion (10) was derived for a general case of non- isotropic dielectrics. Now attention will be restricted to isotropic materials. Assuming small strains and an electric field of high intensity, constitutive relations (4) for an isotropic dielectric can take the form

$$\begin{aligned}&\sigma_k{}^l=\lambda u^m{}_{;m}\delta_k{}^l+\mu\left(u_{k;}{}^l+u^l{}_{;k}\right),\quad t_k{}^l=a_2\delta_k{}^l\phi_{;m}\phi_{;}{}^m+a_5\phi_{;k}\phi^l\\&D_k=-\varepsilon\phi_{;k},\qquad \tau_k{}^l=\varepsilon\left(\phi_{;k}\phi_{;}{}^l-\frac{1}{2}\delta_k{}^l\phi_{;m}\phi_{;}{}^m\right)\end{aligned} \tag{11}$$

where λ, μ, α_2, α_5 are material constants. In this case the electric field is determined as in a rigid body. The stresses $\mathbf{t}$ and $\boldsymbol{\tau}$ are nonlinear with respect to the electric field and play the role of an external load in the equilibrium equations and in the corresponding boundary conditions. In further considerations it is assumed that the dielectric is under the influence of an external electric field and no mechanical load is taken into account.

The construction of the solution of the problem consists of three stages

1) construction of the solution for an electric field,
2) determination of the electric stresses **t**, $\boldsymbol{\tau}$ appearing in the equilibrium equations and the corresponding boundary conditions,
3) construction of the solution describing the displacements generated by the electric field.

The procedure of construction of the solution will be not discussed here in detail. Attention will be restricted to some main problems and the final results will be given.

Let us introduce the polar system of coordinates connected with the Cartesian system by means of the relations

$$x^1 = r\cos\alpha, \quad x^2 = r\sin\alpha$$

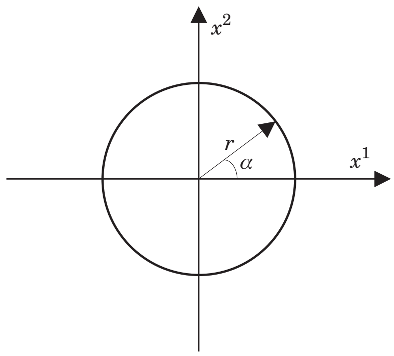

Fig. 2. Polar coordinates

Let us consider the neighbourhood of the crack tip $V_0 : r < r_0, 0 < \alpha < 2\pi$. The domain is bounded by a surface $S_d : r = r_0,\ 0 < \alpha < 2\pi$ and by a segment of the crack surface $S_c :\ r < r_0,\ \ \alpha = \begin{cases} 0 \\ 2\pi \end{cases}$.

The formulation of a boundary value problem requires to define a crack in a dielectric. In the mechanical case the crack is a cut in material and its surface is free from stresses. In the case of an electric field the electric properties of media contained inside the crack are important. It is reasonable to assume that the crack is filled with air, which for a wide range of electric intensities behaves like a vacuum. However, if the electric intensity achieves a ionization value, which for air is about 2–3$\times 10^4$ V/cm, the gas inside the crack behaves like a perfect conductor. Between these two limit cases the crack behaves like a conductor of a finite conductivity. Hence, three cases of cracks are discussed here, i.e. a vacuum crack, a perfectly conducting crack and a crack of finite conductivity. In the case of the vacuum crack and the crack of finite conductivity, the electric field inside the crack is a solution of (3), which in the case considered, when there is no electric charge inside the crack, reduces to the equation $\phi_{;k}{}^{k}=0$. Hence, we define here the crack as a limit case of a wedge of an angle 2δ when $\delta \to 0$. If the electric potentials for the wedge are denoted ϕ^{δ}, the electric intensity for a dielectric with a crack will be given as a limit $\mathbf{E}=-\lim_{\delta\to 0} grad\phi^{\delta}$.

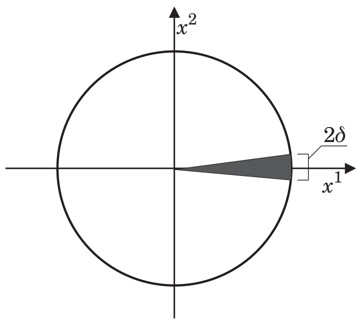

Fig. 3. Wedge modeling a crack segment

The aim is to develop a local solution describing the electric field **E**, the electric stresses **t**, $\boldsymbol{\tau}$ in the domain of a dielectric and on the crack surface, and the displacements **u** and the mechanical stresses $\boldsymbol{\sigma}$ generated by the electric field in a dielectric.

The local solution of the problem has the form of functions which satisfy:

1) the equations in the domain of a dielectric and in the case of an electric field in the domain of the wedge,
2) the corresponding boundary conditions on the crack surface
3) the energy existence condition at the crack tip equivalent to the equalities

$$\lim_{\xi\to 0}\left[\lim_{\delta\to 0}\left\{\int_{\delta}^{2\pi-\delta}\int_{0}^{\xi}\left[\left(\frac{\partial\phi^{\delta}}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial\phi^{\delta}}{\partial\alpha}\right)^2\right]rdrd\alpha+\int_{-\delta}^{\delta}\int_{0}^{\xi}\left[\left(\frac{\partial\breve{\phi}^{\delta}}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial\breve{\phi}^{\delta}}{\partial\alpha}\right)^2\right]rdrd\alpha\right\}\right]=0$$

$$\lim_{\xi\to 0}\int_{0}^{2\pi}\int_{0}^{\xi}\left[\left(\sigma_{rr}+t_{rr}+\tau_{rr}\right)e_{rr}+\left(\sigma_{ra}+t_{r\alpha}+\tau_{r\alpha}\right)e_{ra}+\left(\sigma_{\alpha\alpha}+t_{\alpha\alpha}+\tau_{\alpha\alpha}\right)e_{\alpha\alpha}\right]rdrd\alpha=0,$$

where $\breve{\phi}^{\delta}$ denotes the electric potential inside the wedge,

4) contain such a number of arbitrary constants, which allows to satisfy the boundary conditions on the surface S_d: $\phi(r_0,\alpha)=g(\alpha)$, $\mathbf{u}(r_0,a)=\mathbf{h}(a)$, where $g(a)$, $\mathbf{h}(\alpha)$ are arbitrary analytic functions of the variable α.

The above requirements are sufficient for the uniqueness of the solution.

In the polar system of coordinates the equations for the electric potential reduce to the Laplace equations

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\alpha^2}\right)\begin{pmatrix}\phi^\delta\\ \breve{\phi}^\delta\end{pmatrix}=0 \tag{12}$$

The solution of the Laplace equation takes the form

$$\sum_{n=N}^{\infty} r^{\lambda_n}\left(A_n\cos\lambda_n\alpha+B_n\sin\lambda_n\alpha\right)$$

where λ_n is a sequence determined from the condition of existence of a nontrivial solution satisfying boundary conditions on the crack surface, A_n, B_n are arbitrary constants, some of them determined from the boundary conditions on the crack surface, the other necessary for satisfying the boundary conditions on the surface S_d, N is an integer determined from the energy existence condition.

The following local solution for an electrostatic field has been obtained:

1) the vacuum crack

$$E_r=\sum_{n=1}^{\infty} r^{n-1}\left(A_n{}^v\sin n\alpha+B_n{}^v\cos n\alpha\right),\ \breve{E}_r=\sum_{n=1}^{\infty} r^{n-1}B_n{}^v,$$

$$E_\alpha=\sum_{n=1}^{\infty} r^{n-1}\left(A_n{}^v\cos n\alpha-B_n{}^v\sin n\alpha\right),\ \breve{E}_\alpha=\frac{\varepsilon}{e_0}\sum_{n=1}^{\infty} r^{n-1}nA_n{}^v.$$

2) the perfectly conducting crack

$$E_r=\sum_{n=1}^{\infty} r^{\frac{n}{2}-1}A_n\sin\frac{n}{2}\alpha,\ E_\alpha=\sum_{n=1}^{\infty} r^{\frac{n}{2}-1}A_n\cos\frac{n}{2}\alpha,\ \breve{\mathbf{E}}=\mathbf{0}.$$

3) the conducting crack

$$E_r=\sum_{n=1}^{\infty}\left(r^{n-1}c_n\cos n\alpha+r^{\frac{n}{2}-1}d_n\sin\frac{n}{2}\alpha\right),\quad \breve{E}_r=\sum_{n=1}^{\infty} r^{n-1}c_n,$$

$$E_\alpha=\sum_{n=1}^{\infty}\left(-r^{n-1}c_n\sin n\alpha+r^{\frac{n}{2}-1}d_n\cos\frac{n}{2}\alpha\right),\ \breve{E}_\alpha=0$$

where $A_n{}^v$, $B_n{}^v$, A_n, c_n, d_n are arbitrary constants, which can be determined from the conditions on the boundary S_d of the neighbourhood of the crack tip.

Inserting constitutive equations (11) for the stresses $\boldsymbol{\sigma}$, $\mathbf{t}$, $\boldsymbol{\tau}$ into equilibrium equations (1), the Lamé equations for the displacements are obtained

$$(\lambda+2\mu)\frac{\partial}{\partial r}\left\{\frac{1}{r}\left[\frac{\partial}{\partial r}(ru_r)+\frac{\partial u_\alpha}{\partial \alpha}\right]\right\}-\frac{\mu}{r^2}\frac{\partial}{\partial \alpha}\left[\frac{\partial}{\partial r}(ru_\alpha)-\frac{\partial u_r}{\partial \alpha}\right]=-F_r$$
$$(\lambda+2\mu)\frac{1}{r^2}\frac{\partial}{\partial \alpha}\left[\frac{\partial}{\partial r}(ru_r)+\frac{\partial u_\alpha}{\partial \alpha}\right]+\mu\frac{\partial}{\partial r}\left\{\frac{1}{r}\left[\frac{\partial}{\partial r}(ru_\alpha)-\frac{\partial u_r}{\partial \alpha}\right]\right\}=-F_\alpha \tag{13}$$

where λ, μ are material constants, depend on the electric field.

$$F_r=(2a_2+a_5)\left(E_r\frac{\partial E_r}{\partial r}+E_\alpha\frac{\partial E_\alpha}{\partial r}\right),\quad F_\alpha=(2a_2+a_5)\left(E_r\frac{\partial E_\alpha}{\partial r}-E_\alpha\frac{\partial E_r}{\partial r}\right)$$

Assuming that the series representing the electric field are absolutely convergent, we can multiply them and obtain components of the stresses $\mathbf{t}$ and $\boldsymbol{\tau}$ and the electric terms $\mathbf{F}$ appearing in the Lamé equations in the form of the series:

1) for the vacuum crack

$$\mathbf{t}+\boldsymbol{\tau}=\sum_{n=1}^{\infty} r^{n-1}\mathbf{T}_n{}^v(\alpha),\quad \mathbf{F}=\sum_{n=1}^{\infty} r^{n-1}\mathbf{F}_n{}^v(\alpha),$$

2) for the perfectly conducting crack

$$\mathbf{t}+\boldsymbol{\tau}=\sum_{n=0}^{\infty} r^{\frac{n}{2}-1}\mathbf{T}_n(\alpha),\quad \mathbf{F}=\sum_{n=0}^{\infty} r^{\frac{n}{2}-2}\mathbf{F}_n(\alpha),$$

3) for the conducting crack

$$\mathbf{t}+\boldsymbol{\tau}=\sum_{n=0}^{\infty} r^{\frac{n}{2}-1}\mathbf{T}_n{}^c(\alpha),\quad \mathbf{F}=\sum_{n=0}^{\infty} r^{\frac{n}{2}-2}\mathbf{F}_n{}^c(\alpha).$$

The functions describing the electric stresses have no singularities at the tip of the vacuum crack. At the tip of the conducting and perfectly conducting cracks there are singularities of the order r^{-1}. The admissible singularity of the displacements from the point of view of the energy existence condition is $r^{-1/2}$. If the expressions received in the formal way for the electric stresses are taken into account, the energy condition concerning the energy of coupling will not be satisfied. For this reason a definition of physical electric stresses is introduced.

Definition: The parts of the expressions obtained on a formal way for electric stresses, which generate physical displacements and satisfy the energy existence condition, are called physical electric stresses.

This leads to neglecting the terms of the order r^{-1} in electric stresses and the corresponding terms of the order r^{-2} in **F**, in the case of the conducting and perfectly conducting crack.

The functions representing the local solution for the displacements generated by physical electric stresses are obtained as a series of the form:

1) for the vacuum crack

$$\mathbf{u} = \sum_{n=1}^{\infty} r^{\frac{n}{2}} \mathbf{u}_n(\alpha) + \sum_{n=1}^{\infty} r^n \mathbf{w}_n(\alpha),$$

2) for the conducting and the perfectly conducting crack

$$\mathbf{u} = \sum_{n=1}^{\infty} r^{\frac{n}{2}} \mathbf{u}_n(\alpha) + \sum_{n=1}^{\infty} r^{\frac{n}{2}} \mathbf{w}^c{}_n(\alpha).$$

The first addend of the sum of the series represents a solution of the homogeneous Lamé equations. Each function coefficient $\mathbf{u}_n$ contains two arbitrary constants, which can be determined from the conditions on the boundary of the considered domain S_d. This part of the solution is the same for the vacuum and the conducting cracks. The second addend represents a function satisfying the non-homogeneous Lamé equations. The function coefficients w_n, $w^c{}_n$ depend on the parameters of the electric field in a dielectric. The coefficients denoted symbolically $w^c{}_n$ differ for the two types of the conducting crack. The most important part of the solution from the point of view of the fracture energy are terms of the order $r^{1/2}$. These parts of the solution describe the behaviour of displacements in the vicinity of the crack tip for $r^{<<1}$. The functions are of the following form:

1) the vacuum crack

$$u_r = r^{\frac{1}{2}} \left\{ C_1 \left(\frac{\lambda+5\mu}{\lambda+\mu} \sin\frac{\alpha}{2} + \sin\frac{3}{2}\alpha \right) + D_1 \left(\frac{\lambda+5\mu}{\lambda+\mu} \cos\frac{\alpha}{2} + 3\cos\frac{3}{2}\alpha \right) \right\} + O\left(r^1\right),$$

$$u_\alpha = r^{\frac{1}{2}} \left\{ C_1 \left(\frac{3\lambda+7\mu}{\lambda+\mu} \cos\frac{\alpha}{2} + \cos\frac{3}{2}\alpha \right) - D_1 \left(\frac{3\lambda+7\mu}{\lambda+\mu} \sin\frac{\alpha}{2} + 3\sin\frac{3}{2}\alpha \right) \right\} + O\left(r^1\right).$$

2) the perfectly conducting crack

$$u_r = r^{\frac{1}{2}}\Big[C_1\left(\frac{\lambda+5\mu}{\lambda+\mu}\sin\frac{\alpha}{2}+\sin\frac{3}{2}\alpha\right)+D_1\left(\frac{\lambda+5\mu}{\lambda+\mu}\cos\frac{\alpha}{2}+3\cos\frac{3}{2}\alpha\right)$$
$$+A_1A_2\left\{-\frac{2a_2+a_5}{3\lambda+7\mu}\cos\frac{\alpha}{2}+\frac{1}{\mu}\left[a_5+\varepsilon+\frac{2a_2+a_5}{2}\left(2-\frac{3\lambda+4\mu}{3\lambda+7\mu}-\frac{\lambda+2\mu}{\lambda+5\mu}\right)\right]\cos\frac{3}{2}\alpha\right\}\Big]$$
$$+O\left(r^1\right),$$

$$u_\alpha = r^{\frac{1}{2}}\Big[C_1\left(\frac{3\lambda+7\mu}{\lambda+\mu}\cos\frac{\alpha}{2}+\cos\frac{3}{2}\alpha\right)-D_1\left(\frac{3\lambda+7\mu}{\lambda+\mu}\sin\frac{\alpha}{2}+3\sin\frac{3}{2}\alpha\right)$$
$$+A_1A_2\left\{\frac{2a_2+a_5}{3\lambda+7\mu}\sin\frac{\alpha}{2}-\frac{1}{\mu}\left[a_5+\varepsilon+\frac{2a_2+a_5}{2}\left(2-\frac{3\lambda+4\mu}{3\lambda+7\mu}-\frac{\lambda+2\mu}{\lambda+5\mu}\right)\right]\sin\frac{3}{2}\alpha\right\}\Big]$$
$$+O\left(r^1\right),$$

3) the conducting crack

$$u_r = r^{\frac{1}{2}}\Big[C_1\left(\frac{\lambda+5\mu}{\lambda+\mu}\sin\frac{\alpha}{2}+\sin\frac{3}{2}\alpha\right)+D_1\left(\frac{\lambda+5\mu}{\lambda+\mu}\cos\frac{\alpha}{2}+3\cos\frac{3}{2}\alpha\right)$$
$$+c_1d_1\left[\left(-\frac{a_5+2a_2}{3\lambda+7\mu}\right)\sin\frac{\alpha}{2}+\frac{1}{\mu}\left(2\left(a_5+2a_2\right)\frac{\mu(\lambda+3\mu)}{(3\lambda+7\mu)(\lambda+5\mu)}-a_5-\varepsilon\right)\sin\frac{3}{2}\alpha\right]$$
$$+d_1d_2\left[\frac{a_5+2a_2}{3\lambda+7\mu}\cos\frac{\alpha}{2}+\frac{1}{\mu}\left(\frac{a_5+2a_2}{2}\left(\frac{3\lambda+4\mu}{3\lambda+7\mu}+\frac{\lambda+2\mu}{\lambda+5\mu}\right)+2a_5+2a_2+\varepsilon\right)\right]\cos\frac{3}{2}\alpha\Big]$$
$$+O\left(r^1\right),$$

$$u_\alpha = r^{\frac{1}{2}}\Big[C_1\left(\frac{3\lambda+7\mu}{\lambda+\mu}\cos\frac{\alpha}{2}+\cos\frac{3}{2}\alpha\right)-D_1\left(\frac{3\lambda+7\mu}{\lambda+\mu}\sin\frac{\alpha}{2}+3\sin\frac{3}{2}\alpha\right)$$
$$+c_1d_1\left[\frac{a_5+2a_2}{\lambda+5\mu}\cos\frac{\alpha}{2}+\frac{1}{\mu}\left(2\left(a_5+2a_2\right)\frac{\mu(\lambda+3\mu)}{(3\lambda+7\mu)(\lambda+5\mu)}-a_5-\varepsilon\right)\cos\frac{3}{2}\alpha\right]$$
$$+d_1d_2\left[\frac{a_5+2a_2}{\lambda+5\mu}\sin\frac{\alpha}{2}-\frac{1}{\mu}\left(\frac{a_5+2a_2}{2}\left(\frac{3\lambda+4\mu}{3\lambda+7\mu}+\frac{\lambda+2\mu}{\lambda+5\mu}\right)+2a_5+2a_2+\varepsilon\right)\sin\frac{3}{2}\alpha\right]\Big]$$
$$+O\left(r^1\right).$$

The part of the functions corresponding to the solution of the homogeneous Lamé equations is analogous to the classical Williams solution for a pure mechanical case WILLIAMS (1957). In the case of the vacuum crack, the electric field influences displacements and the mechanical part of the stresses throughout the arbitrary constants C_1, D_1. In the case of the perfectly conducting crack, the appearance of the arbitrary constants A_1, A_2 present in the solution for an electric field in the terms of the order $r^{-1/2}$ and r^0 shows, that the electric field influences directly displacements in the vicinity of the crack tip. Similarly in the case of the conducting crack, where the arbitrary constants c_1, d_1, d_2, connected with the terms of the order $r^{-1/2}$ and r^0 in the solution for the electric field, appear.

Fracture criterion in the Irwin form

The generalized fracture criterion formulated in section 2 is valid for the vacuum crack and the perfectly conducting crack. If a finite conductivity is taken into considerations, the criterion needs reformulation, since thermal effects that should not be neglected are related to the electric current. As it results from experiments, thermal effects take a considerable part in the electromagnetic fracture. This was the reason why the author launched investigations on the cracks of finite conductivity.

If the local solution in the vicinity of the crack tip for an electro-elastic field is inserted into criterion (10), then after integration and application of the limit procedure, a fracture criterion being a generalization of the Irwin criterion for electric fractures is received. In the case of the vacuum crack the form of the criterion is analogous to that in the mechanical case (IRWIN 1957)

$$\frac{\lambda + 2\mu}{4\mu(\lambda+\mu)}\left[\left(K^M{}_I\right)^2 + \left(K^M{}_{II}\right)^2\right] = \gamma$$

where $K^M{}_I$, $K^M{}_{II}$ are mechanical stress intensity coefficients

$$K^M{}_I \equiv \lim_{r\to 0}\sqrt{2\pi r}\,\sigma_{\alpha\alpha}(r,\pi), \quad K^M{}_{II} \equiv \lim_{r\to 0}\sqrt{2\pi r}\,\sigma_{r\alpha}(r,\pi) \qquad (14)$$

In the case of the vacuum crack, the mechanical stress intensity coefficients depend on the arbitrary constants appearing in the local solution for displacements

$$K^M{}_I = 2\mu\sqrt{2\pi}C_1, \;\; K^M{}_{II} = 2\mu\sqrt{2\pi}D_1.$$

In the case of the perfectly conducting crack the criterion is of the form

$$\frac{\lambda+2\mu}{4\mu(\lambda+\mu)}\left[\left(K^{MC}{}_{I}\right)^2+\left(K^{MC}{}_{II}\right)^2\right]+\frac{1}{2\mu}\left[\frac{\lambda+2\mu}{\lambda+\mu}+\frac{5\lambda+13\mu}{8(\lambda+3\mu)}\right]K^{MC}{}_{II}K^{ME}{}_{II}$$
$$+\frac{1}{2\mu}\left(\frac{\lambda+2\mu}{\lambda+\mu}+\frac{\lambda+\mu}{4(\lambda+3\mu)}\right)K^{MC}{}_{II}K^{E}{}_{II}+\frac{5\lambda+13\mu}{16\mu(\lambda+3\mu)}\left(K^{ME}{}_{II}\right)^2$$
$$+\frac{7\lambda+15\mu}{16\mu(\lambda+3\mu)}K^{ME}{}_{II}K^{E}{}_{II}+\frac{\lambda+\mu}{8\mu(\lambda+3\mu)}\left(K^{E}{}_{II}\right)^2+\varepsilon I^2=\gamma.$$

In the case of the perfectly conducting crack, the mechanical stress coefficient $K^{M}{}_{II}$ consists of two parts. The first depends on the arbitrary constant D_1 as in the pure mechanical case and in the case of the vacuum crack, the second depends on the parameters of an electric field in the vicinity of the crack tip

$$K^{M}{}_{II}=K^{MC}{}_{II}+K^{ME}{}_{II},$$
$$K^{MC}{}_{II}=2\mu\sqrt{2\pi}D_1,\quad K^{ME}{}_{II}=\frac{1}{2}\sqrt{2\pi}A_1A_2\left[2(a_5+\varepsilon)+(2a_5+a_5)\left(2-\frac{\lambda+3\mu}{\lambda+5\mu}-\frac{3\lambda+5\mu}{3\lambda+7\mu}\right)\right].$$

The coefficient $K^{E}{}_{II}$ is an electric stress intensity coefficient defined as follows:

$$K^{E}{}_{II}=\lim_{r\to 0}\sqrt{2\pi r}\left[t_{r\alpha}(r,\pi)+\tau_{r\alpha}(r,\pi)\right].$$

From the local solution for an electric field and the constitutive equations it results:

$$K^{E}{}_{II}=-\sqrt{\frac{2}{\pi}}A_1A_2\left(a_5+\varepsilon\right).$$

The symbol I denotes electric field intensity and is here defined as follows:

$$I\equiv-\frac{\sqrt{\pi}}{2}\lim_{r\to 0}r^{-\frac{1}{2}}\phi(r,\pi).$$

In the case of the perfectly conducting crack it depends on the arbitrary constant connected with the singular part of an electric field:

$$I=-\frac{\sqrt{\pi}}{2}A_1.$$

The criterion in the Irwin form is convenient in applications, if intensity coefficients are known. They can be determined experimentally or analytically. A method for analytic determination of intensity coefficients for a semi-infinite crack in an isotropic, homogeneous elastic dielectric has been presented in KURLANDZKA (1982, 1988, 1991). The method is useful in numerical applications. It was formulated for a dynamical case of fields varying in time, but it can be applied to static cases as well.

Conclusions

It results from the above considerations that an electric field influences the fracture of elastic dielectrics. Analysis of the generalized Irwin criterion shows that in the case of the vacuum crack and the perfectly conducting crack, the form of the criterion differs. In the case of the vacuum crack it is analogous to the criterion in the mechanical case. It results from the fact that an electric field and electric stresses have no singularities at the crack tip. An electric field influences the fracture process only through the mechanical stresses which it generates. Then in this case the method consisting of a comparison of stress intensity coefficients with their critical values can be applied for the estimation of the possibility of fracture. Otherwise in the case of the perfectly conducting crack where an electric field influences the fracture not only throughout generated mechanical stresses, but also directly by electric stresses and by itself. The conclusion is that the influence of an electric and electromagnetic field should be taken into account in the fracture process. In a critical state of the crack, when even a small impulse can initiate crack propagation, it can play a decisive role.

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