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ON THE SINGULARITIES AT THE TIPS OF ORTHOTROPIC WEDGES IN PLANE ELASTICITY (PART ONE)

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Abstract

An analytical description of the strength of singularity at the tip of an orthotropic wedge embedded into an infinite two-dimensional elastic orthotropic body was considered. The considerations were restricted to the wedges symmetrically oriented with respect to the axes of orthotropy. Mixed boundary value conditions were assumed, continuity of both, tractions and displacements at the interfaces were demanded. Only singularities of the type $r^{-\lambda}$, where λ is a real number corresponding to finite elastic energy in the vicinity of the wedge tip ($0 \leq \lambda \leq 1$), were taken into considerations.

The order of singularity λ changes with the wedge opening angle φ. Relations $\lambda - \varphi$ for different sets of elastic constants have been studied.

For the case of nearly isotropic materials, two modes of stress distribution with different values of λ: symmetric and skew-symmetric were found. The quantitative results roughly repeated those obtained by the authors for isotropic materials where the symmetries of solutions were assumed in advance.

O RZĘDZIE OSOBLIWOŚCI W OTOCZENIU WIERZCHOŁKA ORTOTROPOWEGO KLINA W PŁASKIM ZAGADNIENIU TEORII SPRĘŻYSTOŚCI (Cz. 1)

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Słowa kluczowe: anizotropia sprężysta, osobliwości pól naprężeń.

S t r e s z c z e n i e

Rozważano opis analityczny rzędu osobliwości w otoczeniu wierzchołka ortotropowego klina zanurzonego w skończonym dwuwymiarowym ortotropowym ciele sprężystym. Rozpatrzono przypadek klina symetrycznego zorientowanego zgodnie z osiami ortotropii. Mieszane warunki brzegowe narzucają ciągłość naprężeń i przemieszczeń na płaszczyznach podziału. Ograniczono się do rozważań rzędu osobliwości $r^{-\lambda}$, odpowiadającego skończonej wartości energii sprężystej w otoczeniu wierzchołka klina ($0 \leq \lambda \leq 1$).

Rząd osobliwości λ zmienia się ze zmianą kąta rozwarcia klina. Zbadano przebiegi zmienności λ przy różnych kombinacjach stałych sprężystych.

Dla przypadku prawie izotropowego wykryto dwa rozkłady naprężeń z różnymi wartościami λ: symetryczne i antysymetryczne. Wyniki ilościowe są bliskie otrzymanym przez autorów dla materiałów izotropowych, gdzie symetria pól naprężeń była założona z góry.

Introduction

The problems of the application of the results of considerations of the singularities at the tips of notches, wedges and cracks in computations, as well as their theoretical importance, have been discussed by numerous authors (NOWACKI 1970), (SEWERYN 1997). Thus this matter will not be discussed in the present paper.

ATKINSON (1997), BLANCO, MARINEZ-ESANOLA and ATKINSON (1998) and WILLIAMS (1952) describe the problems related to stress singularities in the theory of elasticity. Particularly, the elastic stress field near the tip of a sharp angular notch is singular with singularity weaker than that of a crack, that is, the stresses behave like $\sigma \approx r^{-\lambda}$, with $0 < \lambda < \frac{1}{2}$, where r is measured from the notch tip. In isotropic materials, the stress exponent λ depends only on the notch angle; see for example, WILLIAMS (1952), ATKINSON et al. (1988). However, in the case of anisotropic materials, the severity of stress singularity is also dependent upon the properties of the material. For example, BOGY (1968), HEINZELMANN et al. (1994) and BUSH et al (1994) have developed asymptotic methods for angular notches in orthotropic materials, which show a dependence on elastic constants. SEWERYN and MOLSKI (1996) considered elastic stress singularities and generalised stress intensity factors for angular corners under various boundary conditions. BLINOWSKI and ROGACZEWSKI (2000) have confined their attention to the cases when the axis of symmetry of the infinite notch coincides with the axis of orthotropy. BLINOWSKI and WIEROMIEJ (2004) considered the case of mixed boundary conditions for the stress field at the tip of an isotropic wedge embedded in an isotropic medium.

In this paper we shall consider a general case of an orthotropic wedge embedded in an orthotropic medium. In the foregoing sections we shall briefly sketch the way leading to analytic self-similar solutions in polar co-ordinates. We shall also show how to choose the solutions which fulfill the imposed boundary value conditions at contact planes.

In the present Part One we shall confine our attention to the cases when the axis of symmetry of an infinite wedge coincides with the axes of orthotropy in both materials.

We shall take into consideration only the singularities giving rise to finite elastic energy in the vicinity of a sharp corner, i.e. we shall discuss only the values of λ from the interval $\left(0, \frac{1}{2}\right)$. For the time being we shall disregard also possible complex values of λ generating oscillatory solutions of the frequency growing to infinity in the vicinity of the tip.

All these problems, however, can be important for practical computations, particularly if a small plastic zone is expected around the tip. Nevertheless, we shall postpone all the considerations on these topics to the foregoing second part of the present paper.

Basic relations

An analytic description of the strength of singularity will be based on the concept proposed by Blinowski and Ostrowska-Maciejewska (1995), who used a displacement function, helpful in the case of kinematical and mixed boundary value conditions.

Thus we introduce a function $G(x_1, x_2)$ such that the displacement fields $u_1(x_1, x_2)$, $u_2(x_1 x_2)$ in the Cartesian co-ordinates $\{x_1, x_2\}$ associated with the axes of orthotropy can be expressed as follows:

$$\begin{aligned} u_1 &= \frac{1}{\sqrt{E_1 E_2}} \left(\frac{1}{\gamma_1 \gamma_2} G_{,22} + \frac{\gamma_1^2 + \gamma_2^2 - 2\gamma_3^2}{2\gamma_1 \gamma_2} G_{,11} \right)_{,2} \\ u_2 &= \frac{1}{\sqrt{E_1 E_2}} \left(\gamma_1 \gamma_2 G_{,11} + \frac{\gamma_1^2 + \gamma_2^2 - 2\gamma_3^2}{2\gamma_1 \gamma_2} G_{,22} \right)_{,1} \end{aligned} \tag{1}$$

where the comma denotes a partial derivative with respect to the Cartesian co-ordinates mentioned above.

E_1, E_2 – denote Young moduli,

$\gamma_1, \gamma_2, \gamma_3$ – dimensionless elastic constants, fulfilling the following relations:

$$\gamma_1^2\gamma_2^2 = \frac{E_1}{E_2}, \qquad \gamma_1^2 + \gamma_2^2 = 2\left(\frac{E_1}{2\mu} - \nu_{12}\right), \qquad \gamma_3^2 = \frac{E_1}{2\mu}.$$

where υ_{12} is a Poisson ratio,
μ denote a shear modulus.

In the case of isotropic materials we have $\gamma_1 = \gamma_2 = 1$, $\gamma_3^2 = 1 + \nu$.

The displacement function $G(x_1, x_2)$ and the Airy stress function $\Phi(x_1, x_2)$ are related to each other with the following relation:

$$\Phi(x_1, x_2) = G(x_1, x_2)_{,12} \tag{2}$$

Both functions $G(x_1, x_2)$ and $\Phi(x_1, x_2)$ in the Cartesian co-ordinates $\{x_1, x_2\}$, fulfill the following differential equation:

$$\Phi_{,2222} + \left(\gamma_1^2 + \gamma_2^2\right)\Phi_{,1122} + \gamma_1^2\gamma_2^2\Phi_{,1111} = 0, \tag{3}$$

(compare (2.9) in (5)). This equation can be rewritten in the following form:

$$\left(\frac{\partial}{\partial x_2} + i\gamma_1\frac{\partial}{\partial x_1}\right)\left(\frac{\partial}{\partial x_2} - i\gamma_1\frac{\partial}{\partial x_1}\right)\left(\frac{\partial}{\partial x_2} + i\gamma_2\frac{\partial}{\partial x_1}\right)\left(\frac{\partial}{\partial x_2} - i\gamma_2\frac{\partial}{\partial x_1}\right)\Phi(x_1, x_2) = 0, \tag{4}$$

Thus, obviously any differentiable function of any of the following complex variables:

$$\xi_1 = x_1 + i\gamma_1 x_2, \;\; \bar{\xi}_1 = x_1 - i\gamma_1 x_2, \;\; \xi_2 = x_1 + i\gamma_2 x_2, \;\; \bar{\xi}_2 = x_1 - i\gamma_2 x_2 \tag{5}$$

satisfies Eq. (3)

For the case under consideration we can look for the following real displacement function $G(x_1, x_2)$:

$$G(x_1, x_2) = C_1\xi_1^{4-\lambda} + \bar{C}_1\bar{\xi}_1^{4-\lambda} + C_2\xi_2^{4-\lambda} + \bar{C}_2\bar{\xi}_2^{4-\lambda}, \tag{6}$$

where:

$C_1 = A_1 + iB_1,$
$\bar{C}_1 = A_1 - iB_1,$
$C_2 = A_2 + iB_2,$
$\bar{C}_2 = A_2 - iB_2$

and A_1, A_2, B_1, B_2 are arbitrary real numbers.

For the Airy stress function one obtains readily:

$$\Phi(x_1,x_2)=\frac{\partial^2 G}{\partial x_1 \partial x_2}=i(4-\lambda)(3-\lambda)\left[\gamma_1\left(C_1\xi_1^{2-\lambda}-\bar{C}_1\bar{\xi}_1^{2-\lambda}\right)+\gamma_2\left(C_2\xi_2^{2-\lambda}-\bar{C}_2\bar{\xi}_2^{2-\lambda}\right)\right] \quad (7)$$

The stress fields can be expressed as follows:

$$\sigma_{11}=\Phi_{,22}=-i(4-\lambda)(3-\lambda)(2-\lambda)(1-\lambda)\left[\gamma_1^3\left(C_1\xi_1^{-\lambda}-\bar{C}_1\bar{\xi}_1^{-\lambda}\right)+\gamma_2^3\left(C_2\xi_2^{-\lambda}-\bar{C}_2\bar{\xi}_2^{-\lambda}\right)\right] \quad (8)$$

$$\sigma_{22}=\Phi_{,11}=i(4-\lambda)(3-\lambda)(2-\lambda)(1-\lambda)\left[\gamma_1\left(C_1\xi_1^{-\lambda}-\bar{C}_1\bar{\xi}_1^{-\lambda}\right)+\gamma_2\left(C_2\xi_2^{-\lambda}-\bar{C}_2\bar{\xi}_2^{-\lambda}\right)\right] \quad (9)$$

$$\sigma_{12}=-\Phi_{,12}=(4-\lambda)(3-\lambda)(2-\lambda)(1-\lambda)\left[\gamma_1^2\left(C_1\xi_1^{-\lambda}+\bar{C}_1\bar{\xi}_1^{-\lambda}\right)+\gamma_2^2\left(C_2\xi_2^{-\lambda}+\bar{C}_2\bar{\xi}_2^{-\lambda}\right)\right] \quad (10)$$

In the polar co-ordinate system $\{r,\varphi\}$ associated with the observer frame axes, we can write the following expressions for the stress values at radial planes:

$$\begin{aligned}
\sigma_{rr}&=\sigma_{11}\cos^2(\varphi-\psi)+2\sigma_{12}\sin(\varphi-\psi)\cos(\varphi-\psi)+\sigma_{22}\sin^2(\varphi-\psi)\\
\sigma_{\varphi\varphi}&=\sigma_{11}\sin^2(\varphi-\psi)-2\sigma_{12}\sin(\varphi-\psi)\cos(\varphi-\psi)+\sigma_{22}\cos^2(\varphi-\psi)\\
\sigma_{r\varphi}&=(\sigma_{22}-\sigma_{11})\sin(\varphi-\psi)\cos(\varphi-\psi)+\sigma_{12}\left[\cos^2(\varphi-\psi)-\sin^2(\varphi-\psi)\right]
\end{aligned} \quad (11)$$

where φ is an angular co-ordinate, constant along the chosen plane, ψ denote the angle between the axis of the material orthotropy and the axis of the observer frame (Compare Fig. 1).

Substituting relations (8), (9) and (10) into (11) one obtains the following expressions for radial, angular and tangent stresses:

$$\sigma_{rr}=(4-\lambda)(3-\lambda)(2-\lambda)(1-\lambda)\left\{\begin{array}{l}
i\gamma_1\left[\sin(\varphi-\psi)-i\gamma_1\cos(\varphi-\psi)\right]^2 C_1\xi_1^{-\lambda}-\\
-i\gamma_1\left[\sin(\varphi-\psi)+i\gamma_1\cos(\varphi-\psi)\right]^2\bar{C}_1\bar{\xi}_1^{-\lambda}+\\
+i\gamma_2\left[\sin(\varphi-\psi)-i\gamma_2\cos(\varphi-\psi)\right]^2 C_2\xi_2^{-\lambda}-\\
-i\gamma_2\left[\sin(\varphi-\psi)+i\gamma_2\cos(\varphi-\psi)\right]^2\bar{C}_2\bar{\xi}_2^{-\lambda}
\end{array}\right\} \quad (12)$$

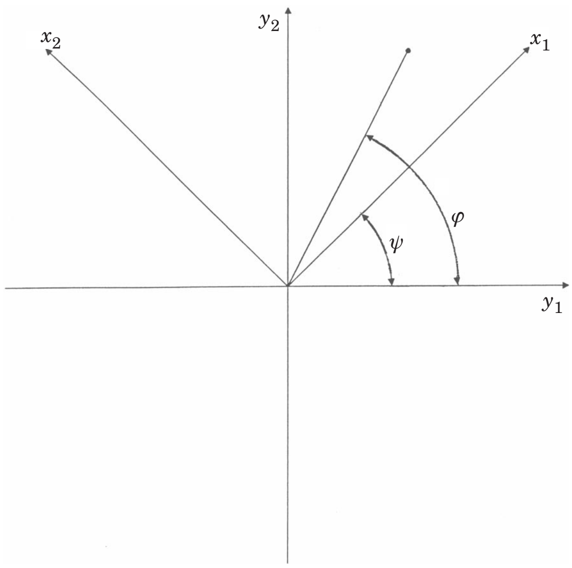

Fig. 1. General scheme of the material orientation:
$\{x_1, x_2\}$ – material co-ordinates associated with the orthotropy axes,
$\{y_1, y_2\}$ – observer Cartesian frame

$$\sigma_{\varphi\varphi} = (4-\lambda)(3-\lambda)(2-\lambda)(1-\lambda)\left\{\begin{array}{l} i\gamma_1\left[\cos(\varphi-\psi)+i\gamma_1\sin(\varphi-\psi)\right]^2 C_1\xi_1^{-\lambda} - \\ -i\gamma_1\left[\cos(\varphi-\psi)-i\gamma_1\sin(\varphi-\psi)\right]^2 \bar{C}_1\bar{\xi}_1^{-\lambda} + \\ +i\gamma_2\left[\cos(\varphi-\psi)+i\gamma_2\sin(\varphi-\psi)\right]^2 C_2\xi_2^{-\lambda} - \\ -i\gamma_2\left[\cos(\varphi-\psi)-i\gamma_2\sin(\varphi-\psi)\right]^2 \bar{C}_2\bar{\xi}_2^{-\lambda} \end{array}\right\} \tag{13}$$

$$\sigma_{r\varphi} = (4-\lambda)(3-\lambda)(2-\lambda)(1-\lambda)\left\{\begin{array}{l} \left[\begin{array}{l} i\gamma_1\left(1+\gamma_1^2\right)\sin(\varphi-\psi)\cos(\varphi-\psi)+ \\ \gamma_1^2\left(\cos^2(\varphi-\psi)-\sin^2(\varphi-\psi)\right) \end{array}\right] C_1\xi_1^{-\lambda} - \\ -\left[\begin{array}{l} i\gamma_1\left(1+\gamma_1^2\right)\sin(\varphi-\psi)\cos(\varphi-\psi)- \\ \gamma_1^2\left(\cos^2(\varphi-\psi)-\sin^2(\varphi-\psi)\right) \end{array}\right] \bar{C}_1\bar{\xi}_1^{-\lambda} + \\ +\left[\begin{array}{l} i\gamma_2\left(1+\gamma_2^2\right)\sin(\varphi-\psi)\cos(\varphi-\psi)+ \\ \gamma_2^2\left(\cos^2(\varphi-\psi)-\sin^2(\varphi-\psi)\right) \end{array}\right] C_2\xi_2^{-\lambda} - \\ -\left[\begin{array}{l} i\gamma_2\left(1+\gamma_2^2\right)\sin(\varphi-\psi)\cos(\varphi-\psi)- \\ \gamma_2^2\left(\cos^2(\varphi-\psi)-\sin^2(\varphi-\psi)\right) \end{array}\right] \bar{C}_2\bar{\xi}_2^{-\lambda} \end{array}\right\} \tag{14}$$

where variables $\xi_1, \bar{\xi}_1, \xi_2, \bar{\xi}_2$ assume the following values:

$$\begin{aligned}
&\xi_1(\varphi,\psi,\gamma_1) = r[\cos(\varphi-\psi) + i\gamma_1 \sin(\varphi-\psi)] \\
&\overline{\xi_1}(\varphi,\psi,\gamma_1) = r[\cos(\varphi-\psi) - i\gamma_1 \sin(\varphi-\psi)] \\
&\xi_2(\varphi,\psi,\gamma_2) = r[\cos(\varphi-\psi)) + i\gamma_2 \sin(\varphi-\psi)] \\
&\overline{\xi_2}(\varphi,\psi,\gamma_2) = r[\cos(\varphi-\psi)) - i\gamma_2 \sin(\varphi-\psi)]
\end{aligned} \tag{15}$$

Powers $4-\lambda$ of the complex variables at expressions for the function $G(x_1, x_2)$ generate singularity $r^{-\lambda}$ in the stress fields.

Substituting the expressions like

$$\begin{aligned}
&C_1\xi_1^{1-\lambda} = (A_1 + iB_1)(\mathrm{Re}\,\xi_1^{1-\lambda} + i\,\mathrm{Im}\,\xi_1^{1-\lambda}) = (A_1\,\mathrm{Re}\,\xi_1^{1-\lambda} - B_1\,\mathrm{Im}\,\xi_1^{1-\lambda}) + \\
&+ i(A_1\,\mathrm{Im}\,\xi_1^{1-\lambda} + B_1\,\mathrm{Re}\,\xi_1^{1-\lambda})
\end{aligned} \tag{16}$$

into (13) and making use of the relation:

$$i\gamma_1[\cos(\varphi-\psi) + i\gamma_1 \sin(\varphi-\psi)]^2 C_1\xi_1^{-\lambda} = \frac{i\gamma_1}{r^2} C_1\xi_1^{2-\lambda} \tag{17}$$

one obtains:

$$\sigma_{\varphi\varphi} = \frac{-2(4-\lambda)(3-\lambda)(2-\lambda)(1-\lambda)}{r^2} \begin{Bmatrix} \gamma_1[A_1\,\mathrm{Im}(\xi_1^{2-\lambda}) + B_1\,\mathrm{Re}(\xi_1^{2-\lambda})] + \\ + \gamma_2[A_2\,\mathrm{Im}(\xi_2^{2-\lambda}) + B_2\,\mathrm{Re}(\xi_2^{2-\lambda})] \end{Bmatrix} \tag{18}$$

Substituting expressions (16) into (14) and making use of the relation:

$$\begin{aligned}
&\{i\gamma_1(1+\gamma_1^2)\sin(\varphi-\psi)\cos(\varphi-\psi) + \gamma_1^2[\cos^2(\varphi-\psi) - \sin^2(\varphi-\psi)]\}C_1\xi_1^{-\lambda} = \\
&= i\gamma_1[\cos(\varphi-\psi) + i\gamma_1 \sin(\varphi-\psi)][\sin(\varphi-\psi) - i\gamma_1 \cos(\varphi-\psi)]C_1\xi_1^{-\lambda} = \\
&= \frac{[\gamma_1^2 \cos(\varphi-\psi) + i\gamma_1 \sin(\varphi-\psi)]}{r} C_1\xi_1^{1-\lambda}
\end{aligned} \tag{19}$$

one can obtain:

$$\sigma_{r\varphi} = \frac{2(4-\lambda)(3-\lambda)(2-\lambda)(1-\lambda)}{r} \begin{Bmatrix} A_1[\gamma_1^2\,\mathrm{Re}(\xi_1^{1-\lambda})\cos(\varphi-\psi) - \gamma_1\,\mathrm{Im}(\xi_1^{1-\lambda})\sin(\varphi-\psi)] - \\ - B_1[\gamma_1^2\,\mathrm{Im}(\xi_1^{1-\lambda})\cos(\varphi-\psi) + \gamma_1\,\mathrm{Re}(\xi_1^{1-\lambda})\sin(\varphi-\psi)] + \\ + A_2[\gamma_2^2\,\mathrm{Re}(\xi_2^{1-\lambda})\cos(\varphi-\psi) - \gamma_2\,\mathrm{Im}(\xi_2^{1-\lambda})\sin(\varphi-\psi)] - \\ - B_2[\gamma_2^2\,\mathrm{Im}(\xi_2^{1-\lambda})\cos(\varphi-\psi) + \gamma_2\,\mathrm{Re}(\xi_2^{1-\lambda})\sin(\varphi-\psi)] \end{Bmatrix} \tag{20}$$

Using relations (1), one can write the Cartesian expressions for displacement fields in the following form:

$$u_1 = \frac{(4-\lambda)(3-\lambda)(2-\lambda)}{\sqrt{E_1 E_2}} \times$$
$$\times \left\{ \frac{-\gamma_1^2+\gamma_2^2-2\gamma_3^2}{2\gamma_2} i\left(C_1 \xi_1^{1-\lambda} - \overline{C}_1 \overline{\xi}_1^{\,1-\lambda} \right) + \frac{\gamma_1^2-\gamma_2^2-2\gamma_3^2}{2\gamma_1} i\left(C_2 \xi_2^{1-\lambda} - \overline{C}_2 \overline{\xi}_2^{\,1-\lambda} \right) \right\} \tag{21}$$

$$u_2 = \frac{(4-\lambda)(3-\lambda)(2-\lambda)}{\sqrt{E_1 E_2}} \times$$
$$\times \left\{ -\gamma_1 \frac{\gamma_1^2-\gamma_2^2-2\gamma_3^2}{2\gamma_2} \left(C_1 \xi_1^{1-\lambda} + \overline{C}_1 \overline{\xi}_2^{\,1-\lambda} \right) - \gamma_2 \frac{-\gamma_1^2+\gamma_2^2-2\gamma_3^2}{2\gamma_1} \left(C_2 \xi_2^{1-\lambda} + \overline{C}_2 \overline{\xi}_2^{\,1-\lambda} \right) \right\}$$

Displacement fields in the polar co-ordinates (r,θ) can be expressed in Cartesian terms as follows.

$$\begin{aligned} u_r &= u_1 \cos\theta + u_2 \sin\theta \\ u_\theta &= -u_1 \sin\theta + u_2 \cos\theta \end{aligned} \tag{22}$$

Substituting expressions (22) into (21) and using the relation (16) one can write expressions for displacements in the polar co-ordinates as follows:

$$u_r = -\frac{(4-\lambda)(3-\lambda)(2-\lambda)}{\sqrt{E_1 E_2}}$$
$$\left\{ \begin{aligned} & A_1 \left[\frac{-\gamma_1^2+\gamma_2^2-2\gamma_3^2}{\gamma_2} \operatorname{Im} \xi_1^{1-\lambda} \cos(\varphi-\psi) + \gamma_1 \frac{\gamma_1^2-\gamma_2^2-2\gamma_3^2}{\gamma_2} \operatorname{Re} \xi_1^{1-\lambda} \sin(\varphi-\psi) \right] + \\ & + B_1 \left[\frac{-\gamma_1^2+\gamma_2^2-2\gamma_3^2}{\gamma_2} \operatorname{Re} \xi_1^{1-\lambda} \cos(\varphi-\psi) - \gamma_1 \frac{\gamma_1^2-\gamma_2^2-2\gamma_3^2}{\gamma_2} \operatorname{Im} \xi_1^{1-\lambda} \sin(\varphi-\psi) \right] + \\ & + A_2 \left[\frac{\gamma_1^2-\gamma_2^2-2\gamma_3^2}{\gamma_1} \operatorname{Im} \xi_2^{1-\lambda} \cos(\varphi-\psi) + \gamma_2 \frac{-\gamma_1^2+\gamma_2^2-2\gamma_3^2}{\gamma_1} \operatorname{Re} \xi_2^{1-\lambda} \sin(\varphi-\psi) \right] + \\ & + B_2 \left[\frac{\gamma_1^2-\gamma_2^2-2\gamma_3^2}{\gamma_1} \operatorname{Re} \xi_2^{1-\lambda} \cos(\varphi-\psi) - \gamma_2 \frac{-\gamma_1^2+\gamma_2^2-2\gamma_3^2}{\gamma_1} \operatorname{Im} \xi_2^{1-\lambda} \sin(\varphi-\psi) \right] \end{aligned} \right\} \tag{23}$$

$$u_\varphi = \frac{(4-\lambda)(3-\lambda)(2-\lambda)}{\sqrt{E_1 E_2}}$$

$$\left\{\begin{array}{l} A_1\left[\dfrac{-\gamma_1^2+\gamma_2^2-2\gamma_3^2}{\gamma_2}\operatorname{Im}\xi_1^{1-\lambda}\sin(\varphi-\psi)-\gamma_1\dfrac{\gamma_1^2-\gamma_2^2-2\gamma_3^2}{\gamma_2}\operatorname{Re}\xi_1^{1-\lambda}\cos(\varphi-\psi)\right]+ \\ +B_1\left[\dfrac{-\gamma_1^2+\gamma_2^2-2\gamma_3^2}{\gamma_2}\operatorname{Re}\xi_1^{1-\lambda}\sin(\varphi-\psi)+\gamma_1\dfrac{\gamma_1^2-\gamma_2^2-2\gamma_3^2}{\gamma_2}\operatorname{Im}\xi_1^{1-\lambda}\cos(\varphi-\psi)\right]+ \\ +A_2\left[\dfrac{\gamma_1^2-\gamma_2^2-2\gamma_3^2}{\gamma_1}\operatorname{Im}\xi_2^{1-\lambda}\sin(\varphi-\psi)-\gamma_2\dfrac{-\gamma_1^2+\gamma_2^2-2\gamma_3^2}{\gamma_1}\operatorname{Re}\xi_2^{1-\lambda}\cos(\varphi-\psi)\right]+ \\ +B_2\left[\dfrac{\gamma_1^2-\gamma_2^2-2\gamma_3^2}{\gamma_1}\operatorname{Re}\xi_2^{1-\lambda}\sin(\varphi-\psi)+\gamma_2\dfrac{-\gamma_1^2+\gamma_2^2-2\gamma_3^2}{\gamma_1}\operatorname{Im}\xi_2^{1-\lambda}\cos(\varphi-\psi)\right] \end{array}\right\} \quad (24)$$

Formulation of the boundary value problem

As already mentioned, in the present paper we shall focus our attention on the mixed boundary value problem assuming continuity of displacement field, as well as the normal and tangential stresses at interfaces of an orthotropic wedge embedded in an orthotropic medium. We shall consider cases when the axis of symmetry of an infinite wedge coincides with the axes of orthotropy. All quantities in the domain of the first material will be noted with the Roman number one (I), while for the other material the Roman number two (II) will be used (Fig. 2).

A complete set of boundary value conditions must be taken as follows:

$$\left.\begin{array}{l} u^{I}_{rI,II}-u^{II}_{rI,II}=0 \\ u^{I}_{\varphi I,II}-u^{II}_{\varphi I,II}=0 \end{array}\right\} \text{ – displacement continuity for I-II interface}$$

$$\left.\begin{array}{l} \sigma^{I}_{\varphi\varphi I,II}-\sigma^{II}_{\varphi\varphi I,II}=0 \\ \sigma^{I}_{r\varphi I,II}-\sigma^{II}_{r\varphi I,II}=0 \end{array}\right\} \text{ – traction continuity for I-II interface} \quad (25)$$

$$\left.\begin{array}{l} u^{I}_{rII,I}-u^{II}_{rII,I}=0 \\ u^{I}_{\varphi II,I}-u^{II}_{\varphi II,I}=0 \end{array}\right\} \text{ – displacement continuity for II-I interface}$$

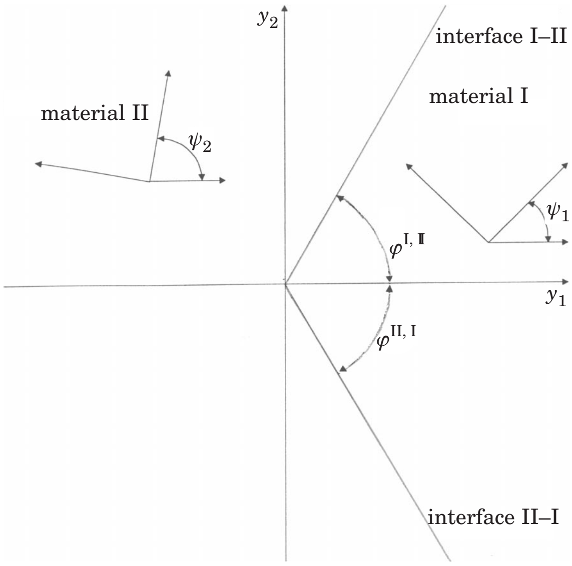

Fig. 2. Orientation of the orthotropic wedge embedded in the orthotropic medium with respect to the observer frame axes $\{y_1, y_2\}$

$$\left.\begin{array}{l} \sigma^{I}_{\varphi\varphi II,I} - \sigma^{II}_{\varphi\varphi II,I} = 0 \\ \sigma^{I}_{r\varphi II,I} - \sigma^{II}_{r\varphi II,I} = 0 \end{array}\right\} \text{ – traction continuity for II-I interface}$$

Conditions (25) and relations (18), (20), (25) generate the following homogenous system of eight equations for determination of the unknown multipliers A_1^I, B_1^I, A_2^I, B_2^I, A_1^{II}, B_1^{II}, A_2^{II}, B_2^{II}.

$$\underline{\mathbf{A}}\ \underline{\mathbf{x}} = 0 \tag{26}$$

where $\underline{\mathbf{x}} = \begin{bmatrix} A_1^I \\ B_1^I \\ A_2^I \\ B_2^I \\ A_1^{II} \\ B_1^{II} \\ A_2^{II} \\ B_2^{II} \end{bmatrix}$, Regarding the terms of the matrix **A** – see Appendix.

For numerical calculations, the values of Young moduli and γ_1 constants were assumed to be known for both materials, the remaining material constants were calculated as follows:

$$\gamma_2 = \frac{\sqrt{\frac{E_1}{E_2}}}{\gamma_1} \tag{27}$$

$$\gamma_3 = \sqrt{\gamma_1 \gamma_2 k + \frac{\left(\gamma_1^2 + \gamma_2^2\right)}{2}} \tag{28}$$

where $0 \le k \le 1$ is a material constant[1]. Its value must be independently assumed.

Values of l are to be determined from the condition

$$\det \mathbf{A} = 0 \tag{29}$$

The calculated numeric values of l can be then substituted into appropriate expressions for the components of the matrix A, and their numeric values, corresponding to subsequent values of λ, can be calculated. Using a standard procedure for determining eigenvectors of the matrix A corresponding to null eigenvalues, one can find components of normalised vectors **x** representing the sets of unknown multipliers

$A_1^I,\ B_1^I,\ A_2^I,\ B_2^I,\ A_1^{II},\ B_1^{II},\ A_2^{II},\ B_2^{II}$.

Preliminary results and conclusions

Plots showing the change of the order of singularity λ versus the wedge opening angle φ, for both symmetric and skew-symmetric cases and for different values of the elastic constants are shown in Figs. 4–8.

In Figs. 4 and 5 one can see the case of nearly isotropic materials. The results are quite similar to those obtained by the BLINOWSKI and WIEREMIEJ (2004) for the isotropic case (cf. Fig. 3.) Let us note, however, that in the present case the symmetry and skew symmetry of stress fields were obtained as a result of the imposed general boundary conditions, while in (BLINOWSKI, WIEROMIEJ 2004) the symmetries were assumed in advance.

[1]It is not difficult to notice that: if $\nu_{12} > 0$ then $\gamma_1^2 + \gamma_2^2 \le 2\gamma_3^2 \le \left(\gamma_2 + \gamma_2\right)^2$, thus $2\gamma_3^2 = \gamma_1^2 - \gamma_2^2 + k2\gamma_1^2\gamma_2^2$

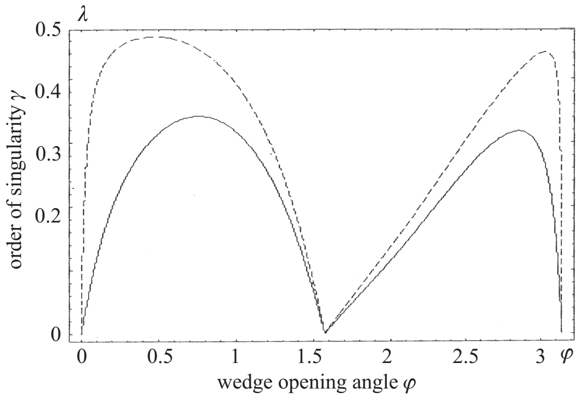

Fig. 3. Isotropic materials, the order of singularity λ versus the opening angle φ (symmetric case) for $E^{I}=1, E^{II}=10$ (solid curve) and for $E^{I}=1, E^{II}=100$ (dashed curve) equal Poisson ratio for both materials $\nu_1=\nu_2=\frac{1}{4}$

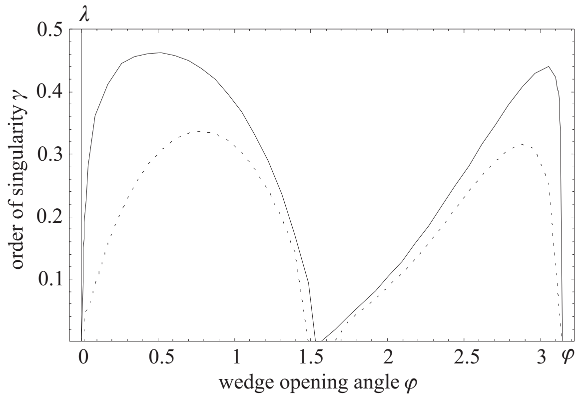

Fig. 4. Order of singularity versus the opening angle φ (symmetric mode):

solid curve: for the Young moduli of the first material $E_1^{I}=0.11,\ E_2^{I}=0.09$ and $E_1^{II}=11,\ E_2^{II}=9$, for the other material.

dashed curve: $E_1^{I}=0.11,\ E_2^{I}=0.09$ for the first and $E_1^{II}=1.1,\ E_2^{II}=0.9$ for the other material, dimensionless constants: $\gamma_1^{I}=\gamma_1^{II}=1.2$, $k^{I}=k^{II}=0.5$

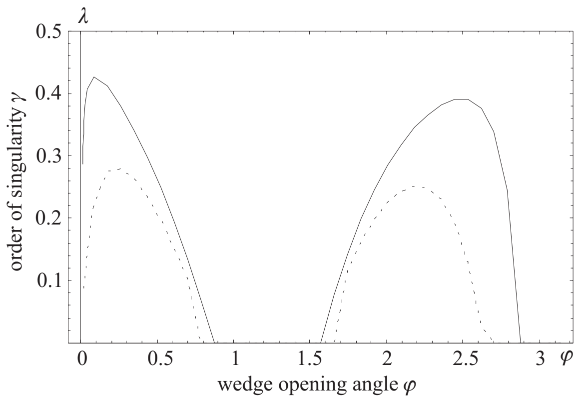

Fig. 5. Order of singularity λ versus the opening angle φ for the skew-symmetric mode and the same materials as in Fig. 4

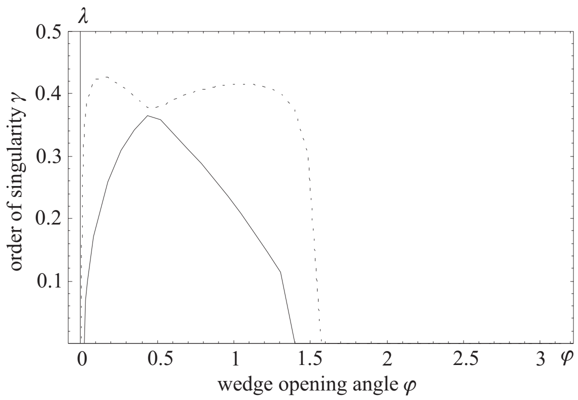

Fig. 6. Order of singularity λ versus the opening angle φ for $E_1^I = 100,\ E_2^I = 1$ and $E_1^{II} = 1,\ E_2^{II} = 100$ (solid and dashed line for both symmetric modes): $\gamma_1^I = \gamma_1^{II} = 1.2$ and $k^I = k^{II} = 0.5$

Quite an interesting case is shown in Figs. 6–8, where the wedge is made of the same material as the matrix merely rotated by $\frac{\pi}{2}$. One can see two symmetric results and one skew-symmetric one for the ratio of Young moduli E_1 / E_2 equal to 100. For a lower ratio $E_1 / E_2 = 10$ only symmetric solutions can be observed.

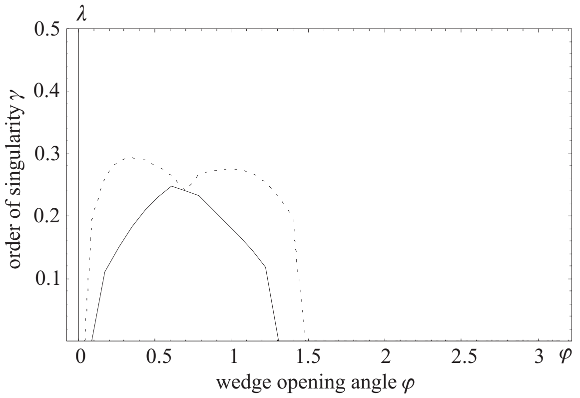

Fig. 7. The same situation as in Fig 6 for lower moduli $E_1^I = 10,\ E_2^I = 1$ and

$E_1^{II} = 1,\ E_2^{II} = 10$, $\gamma_1^I = \gamma_1^{II} = 1.2$, $k^I = k^{II} = 0.5$

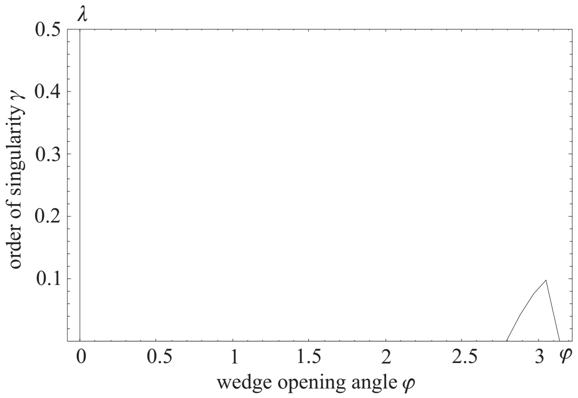

Fig. 8. Order of singularity λ versus the opening angle φ, the skew-symmetric mode for the same materials as in Fig. 6

The last statement does not remain valid for different values of dimensionless constantans, e.g. taking $\gamma_1^I = \gamma_1^{II} = 0.5$, $k^I = k^{II} = 0.1$ one obtains skew-symmetric solutions both for $E_1 / E_2 = 10$ and $E_1 / E_2 = 100$.

Thus, one can see that the method applied by the authors to simple cases of the problem yields reasonable and expected results compatible with the earlier results for the isotropic case. Some results are qualitatively new, e.g. the presence in multiple symmetric solutions.

More detailed studies of the subject, including the investigations of the cases of arbitrary symmetry, need some refinements in the formulation of boundary conditions and will be postponed to Part Two of the present paper.

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Appendix

For reference, we shall expose here explicit expressions for the components of the matrix **A**. For the sake of brevity we shall subdivide the matrix into four square submatrices, as follows:

$$\underline{\mathbf{A}} = \left[\begin{array}{c|c} \mathbf{A}^1 & \mathbf{A}^2 \\ \hline \mathbf{A}^3 & \mathbf{A}^4 \end{array}\right]$$

For the submatrix $\mathbf{A}^1$ one can write:

$$A_{11} = -\left[\frac{-\left(\gamma_1^I\right)^2 + \left(\gamma_2^I\right)^2 - 2\left(\gamma_3^I\right)^2}{\gamma_2^I} Ir\left(\varphi^{I,II} - \psi^I, 1-\lambda, \gamma_1^I\right)\cos\left(\varphi^{I,II} - \psi^I\right) + \gamma_1^I \frac{\left(\gamma_1^I\right)^2 - \left(\gamma_2^I\right)^2 - 2\left(\gamma_3^I\right)^2}{\gamma_2^I} Rr\left(\varphi^{I,II} - \psi^I, 1-\lambda, \gamma_1^I\right)\sin\left(\varphi^{I,II} - \psi^I\right)\right] / E^I,$$

$$A_{12} = -\left[\frac{-\left(\gamma_1^I\right)^2 + \left(\gamma_2^I\right)^2 - 2\left(\gamma_3^I\right)^2}{\gamma_2^I} Rr\left(\varphi^{I,II} - \psi^I, 1-\lambda, \gamma_1^I\right)\cos\left(\varphi^{I,II} - \psi^I\right) - \gamma_1^I \frac{\left(\gamma_1^I\right)^2 - \left(\gamma_2^I\right)^2 - 2\left(\gamma_3^I\right)^2}{\gamma_2^I} Ir\left(\varphi^{I,II} - \psi^I, 1-\lambda, \gamma_1^I\right)\sin\left(\varphi^{I,II} - \psi^I\right)\right] / E^I,$$

$$A_{13} = -\left[\frac{\left(\gamma_1^I\right)^2 - \left(\gamma_2^I\right)^2 - 2\left(\gamma_3^I\right)^2}{\gamma_1^I} Ir\left(\varphi^{I,II} - \psi^I, 1-\lambda, \gamma_2^I\right)\cos\left(\varphi^{I,II} - \psi^I\right) + \gamma_2^I \frac{-\left(\gamma_1^I\right)^2 + \left(\gamma_2^I\right)^2 - 2\left(\gamma_3^I\right)^2}{\gamma_1^I} Rr\left(\varphi^{I,II} - \psi^I, 1-\lambda, \gamma_2^I\right)\sin\left(\varphi^{I,II} - \psi^I\right)\right] / E^I,$$

$$A_{14} = -\left[\frac{\left(\gamma_1^I\right)^2 - \left(\gamma_2^I\right)^2 - 2\left(\gamma_3^I\right)^2}{\gamma_1^I} Rr\left(\varphi^{I,II} - \psi^I, 1-\lambda, \gamma_2^I\right)\cos\left(\varphi^{I,II} - \psi^I\right) - \gamma_2^I \frac{-\left(\gamma_1^I\right)^2 + \left(\gamma_2^I\right)^2 - 2\left(\gamma_3^I\right)^2}{\gamma_1^I} Ir\left(\varphi^{I,II} - \psi^I, 1-\lambda, \gamma_2^I\right)\sin\left(\varphi^{I,II} - \psi^I\right)\right] / E^I,$$

$$A_{21} = \left[\frac{-\left(\gamma_1^I\right)^2 + \left(\gamma_2^I\right)^2 - 2\left(\gamma_3^I\right)^2}{\gamma_2^I} Ir\left(\varphi^{I,II} - \psi^I, 1-\lambda, \gamma_1^I\right)\sin\left(\varphi^{I,II} - \psi^I\right) - \gamma_1^I \frac{\left(\gamma_1^I\right)^2 - \left(\gamma_2^I\right)^2 - 2\left(\gamma_3^I\right)^2}{\gamma_2^I} Rr\left(\varphi^{I,II} - \psi^I, 1-\lambda, \gamma_1^I\right)\cos\left(\varphi^{I,II} - \psi^I\right)\right] / E^I,$$

$$A_{22} = \left[\frac{-\left(\gamma_1^I\right)^2 + \left(\gamma_2^I\right)^2 - 2\left(\gamma_3^I\right)^2}{\gamma_2^I} Rr\left(\varphi^{I,II} - \psi^I, 1-\lambda, \gamma_1^I\right)\sin\left(\varphi^{I,II} - \psi^I\right) + \gamma_1^I \frac{\left(\gamma_1^I\right)^2 - \left(\gamma_2^I\right)^2 - 2\left(\gamma_3^I\right)^2}{\gamma_2^I} Ir\left(\varphi^{I,II} - \psi^I, 1-\lambda, \gamma_1^I\right)\cos\left(\varphi^{I,II} - \psi^I\right)\right] / E^I,$$

$$A_{23}=\left[\frac{\left(\gamma_1^I\right)^2-\left(\gamma_2^I\right)^2-2\left(\gamma_3^I\right)^2}{\gamma_1^I}Ir\left(\varphi^{I,II}-\psi^I,1-\lambda,\gamma_2^I\right)\sin\left(\varphi^{I,II}-\psi^I\right)-\gamma_2^I\frac{-\left(\gamma_1^I\right)^2+\left(\gamma_2^I\right)^2-2\left(\gamma_3^I\right)^2}{\gamma_1^I}Rr\left(\varphi^{I,II}-\psi^I,1-\lambda,\gamma_2^I\right)\cos\left(\varphi^{I,II}-\psi^I\right)\right]/E^I,$$

$$A_{24}=\left[\frac{\left(\gamma_1^I\right)^2-\left(\gamma_2^I\right)^2-2\left(\gamma_3^I\right)^2}{\gamma_1^I}Rr\left(\varphi^{I,II}-\psi^I,1-\lambda,\gamma_2^I\right)\sin\left(\varphi^{I,II}-\psi^I\right)+\gamma_2^I\frac{-\left(\gamma_1^I\right)^2+\left(\gamma_2^I\right)^2-2\left(\gamma_3^I\right)^2}{\gamma_1^I}Ir\left(\varphi^{I,II}-\psi^I,1-\lambda,\gamma_2^I\right)\cos\left(\varphi^{I,II}-\psi^I\right)\right]/E^I,$$

$$A_{31}=-\gamma_1^I Ir\left(\varphi^{I,II}-\psi^I,2-\lambda,\gamma_1^I\right),$$

$$A_{32}=-\gamma_1^I Rr\left(\varphi^{I,II}-\psi^I,2-\lambda,\gamma_1^I\right),$$

$$A_{33}=-\gamma_2^I Ir\left(\varphi^{I,II}-\psi^I,2-\lambda,\gamma_2^I\right),$$

$$A_{34}=-\gamma_2^I Rr\left(\varphi^{I,II}-\psi^I,2-\lambda,\gamma_2^I\right),$$

$$A_{41}=\left[\left(\gamma_1^I\right)^2 Rr\left(\varphi^{I,II}-\psi^I,1-\lambda,\gamma_1^I\right)\cos\left(\varphi^{I,II}-\psi^I\right)-\gamma_1^I Ir\left(\varphi^{I,II}-\psi^I,1-\lambda,\gamma_1^I\right)\sin\left(\varphi^{I,II}-\psi^I\right)\right],$$

$$A_{42}=-\left[\left(\gamma_1^I\right)^2 Ir\left(\varphi^{I,II}-\psi^I,1-\lambda,\gamma_1^I\right)\cos\left(\varphi^{I,II}-\psi^I\right)+\gamma_1^I Rr\left(\varphi^{I,II}-\psi^I,1-\lambda,\gamma_1^I\right)\sin\left(\varphi^{I,II}-\psi^I\right)\right],$$

$$A_{43}=\left[\left(\gamma_2^I\right)^2 Rr\left(\varphi^{I,II}-\psi^I,1-\lambda,\gamma_2^I\right)\cos\left(\varphi^{I,II}-\psi^I\right)-\gamma_2^I Ir\left(\varphi^{I,II}-\psi^I,1-\lambda,\gamma_2^I\right)\sin\left(\varphi^{I,II}-\psi^I\right)\right],$$

$$A_{44}=-\left[\left(\gamma_2^I\right)^2 Ir\left(\varphi^{I,II}-\psi^I,1-\lambda,\gamma_2^I\right)\cos\left(\varphi^{I,II}-\psi^I\right)+\gamma_2^I Rr\left(\varphi^{I,II}-\psi^I,1-\lambda,\gamma_2^I\right)\sin\left(\varphi^{I,II}-\psi^I\right)\right],$$

The elements of the remaining submatrices can be obtained easily by the following modifications of the matrix $\mathbf{A}^1$:

- For the matrix $\mathbf{A}^2$ the variables E^I ψ^I, γ_1^I, γ_2^I should be replaced with E^{II} ψ^{II}, $\gamma_1^{II}, \gamma_2^{II}$ and the signs of all terms should be changed,
- The matrix $\mathbf{A}^3$ can be obtained from $\mathbf{A}^1$ by replacing the variable $\varphi^{I,II}$ with $\varphi^{II,I} + 2\pi$,
- For $\mathbf{A}^4$ one should replace the variable $\varphi^{I,II}$ with $\varphi^{II,I}$ in $\mathbf{A}^2$.